CALIFORNIA INSTITUTE OF TECHNOLOGY

SUBJECT: A collection of formulas useful in particle tracking

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I have managed to collect in one place a set of useful formulas relevant to charged particle tracking in a solenoidal field. This is the arithmetic used in the programs TRKSIM (ASLUND) and SITRKS. It is mainly just cylindrical geometry. This note is not meant to be pedagogical, so it is short on explanations; use it as a reference.

The bible is George Trilling's note LBL-TG-301 (1978), which I enclose as an appendix. Some of the approximate formulas come from R. Gluckstern, NIM 24 (1963) 381.

I include sections on track fitting, estimates of track parameter resolution, the effects of multiple scattering, and "segment finding" in pattern recognition with superlayers. Some of this stuff is only appropriate for drift chambers, or for silicon detectors, or for barrel detectors or endcap detectors.

Unless otherwise specified, we assume a uniform axial magnetic field.

TRACK AND CIRCLE PARAMETERS

Track trajectories in solenoidal detectors are described as piecewise helices. Piecewise, because multiple scattering and non-uniform B-field will result in a deviation from a helix. Nonetheless, we can describe a track trajectory using parameters appropriate for a helix, with axis parallel to the $\hat{z}$ direction (the direction of the B-field). I will define three different sets of track parameters, useful for different situations.

<u>Track bank parameters:</u> for track of charge q (useful for physics analysis):

$$(q, \mathrm{p}_x, \mathrm{p}_y, \mathrm{p}_z, x_b, y_b, z_b)$$

where all parameters are conventionally evaluated at the distance of closest approach of the track helix to the $\hat{z}$ axis, that is, the origin in the xy plane (DCAO); so $\sqrt{x_b^2 + y_b^2} \equiv b$

<u>Track circle parameters,</u> for track at point i, position (x_i, y_i, z_i), momentum (p_x, p_y, p_z) (figure 1); (useful for quick "global" calculations):

$$(q, x_c, y_c, r_c, s, z_b)$$

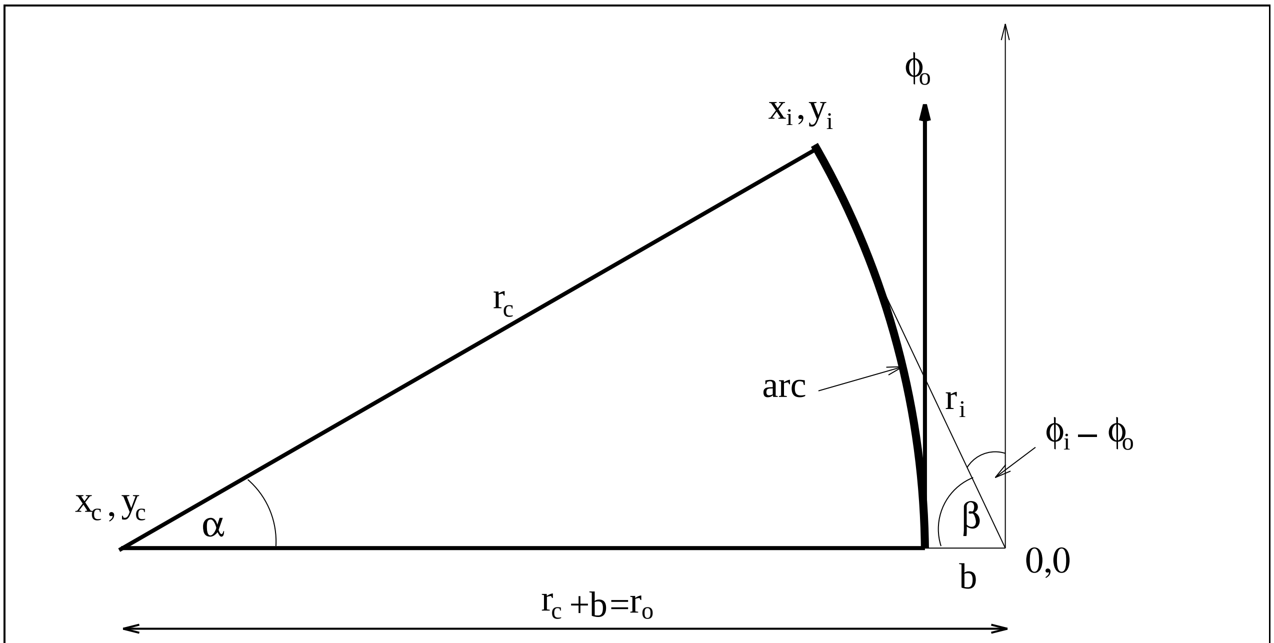

Figure 1: Track arc in xy plane from DCAO to x_i, y_i.

- Transverse momentum $p_t = \sqrt{p_x^2 + p_y^2} = 0.3Br_c$, $\quad B$ in Tesla
- Slope $s \equiv \tan\lambda \equiv p_z/p_t$
- X position of center of circle $x_c = x_i + qp_y/(0.3B)$

  Y position of center of circle $y_c = y_i - qp_x/(0.3B)$

  Distance of center of circle to origin $= r_0 = \sqrt{x_c^2 + y_c^2} \equiv r_c + b$

Track fit parameters (useful for precise local calculations): $(b, \phi_0, \kappa, z_b, s)$

- ϕ at DCAO: $\phi_0 = \mathrm{atan2}(p_y, p_x)$
- curvature $\kappa = -(0.3Bq)/(2p_t) = -q/(2r_c)$ (sometimes one defines $k = q/p_t$)
- distance of closest approach of track to origin (DCAO) is

$$b = \sqrt{x_c^2 + y_c^2} - r_c \equiv r_0 - r_c$$

  Note that b is a signed quantity; but it is *not* the same sign that one would get with the definition $b' = x_b \sin\phi_0 - y_b \cos\phi_0 = qb$; *nor* is it the same sign one would use for a lifetime impact parameter analysis (which depends on the jet direction). Note also that if $b \ll r_c$ the above calculation requires double precision arithmetic.

Track swim. Given track parameters at a point (x_i, y_i, z_i), *e.g.*, at a Monte Carlo vertex, what are the track parameters at another point, *e.g.*, the DCAO? First, calculate circle parameters. Then:

- at DCAO, extrapolation of track is

  $x_b = (r_0 - r_c) \times x_c/r_0$

  $y_b = (r_0 - r_c) \times y_c/r_0$

- What is z_b? We must calculate the xy path length, $r_c\alpha$.
  The turning angle α in the xy plane, between two points on the circle (x_i, y_i) and (x_b, y_b), chord $= c$ (see figure 2), is $\sin(\alpha/2) = c/2r_c$, as long as $\alpha \in [-\pi/2, \pi/2]$.

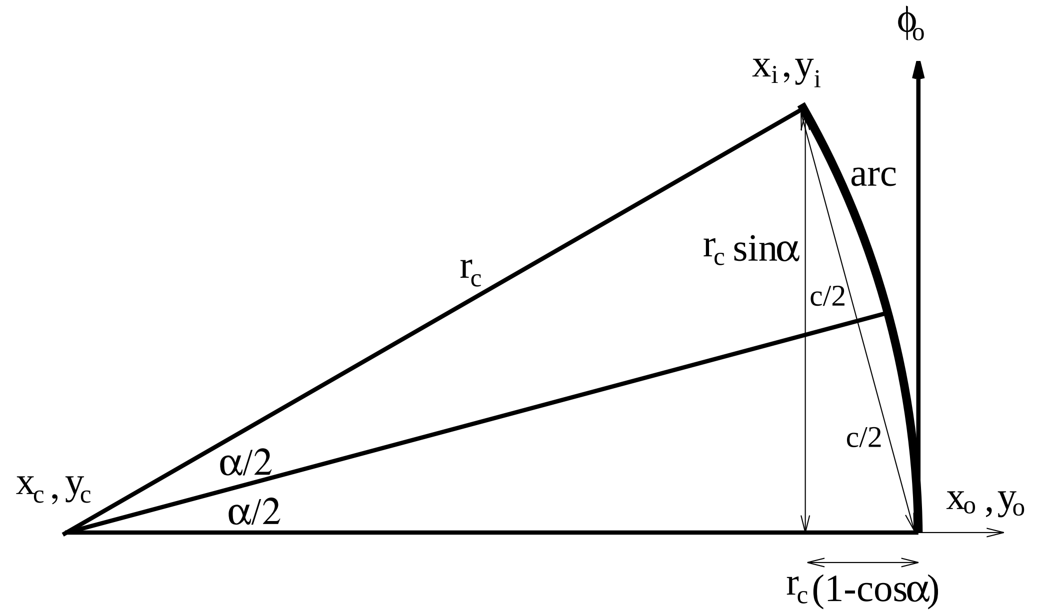

Figure 2: Propagating along an arc in xy plane.

- Another way to get the turning angle α from (x_i, y_i) to the DCAO is to define two unit vectors from figure 1:

$$\hat{r}_{ic} = (x_i - x_c, y_i - y_c)/r_c = (-qp_y/p_t, qp_x/p_t), \qquad \text{and}$$

$$\hat{r}_{0c} = (0 - x_c, 0 - y_c)/r_0 = -(x_i, y_i)/r_0 + (qp_y, -qp_x)/(0.3Br_0)$$

where (p_x, p_y) are evaluated at (x_i, y_i). We then have:

$$\cos\alpha = \hat{r}_{0c} \cdot \hat{r}_{ic} = (r_0^2 - x_i x_c - y_i y_c)/(r_0 r_c) = (qx_i p_y - qy_i p_x - r_c p_t)/(r_0 p_t)$$

$$\sin\alpha = q\hat{r}_{0c} \times \hat{r}_{ic} = q(x_i y_c - y_i x_c)/(r_0 r_c) = -(x_i p_x + y_i p_y)/(r_0 p_t)$$

$$\alpha = \mathrm{atan2}(\sin\alpha, \cos\alpha) \in [0, 2\pi]$$

For loopers, add more multiples of 2π.

- Once we have the xy turning angle α, either from the calculated chord length c or the above geometry, we can then propagate from (x_i, y_i, z_i) to (x_b, y_b, z_b) (or from any pair of points) (see figure 2):

$$\begin{aligned} x_b =& x_i + (\sin\alpha \cdot p_x + q(1-\cos\alpha)\cdot p_y)/(0.3B) \\ y_b =& y_i + (\sin\alpha \cdot p_y - q(1-\cos\alpha)\cdot p_x)/(0.3B) \\ z_b =& z_i + \alpha \cdot p_z/(0.3B) \end{aligned}$$

- and we can evaluate the track parameters at that point:

$$p'_x = q\cdot 0.3B\cdot(y_b - y_c); \qquad p'_y = -q\cdot 0.3B\cdot(x_b - x_c); \qquad p'_z = p_z$$

$$\phi_0 = \mathrm{atan2}(p_y, p_x); \qquad \kappa = -0.3Bq/(2p_t); \qquad s = p_z/p_t; \qquad l_{xyz} = \sqrt{1+s^2}r_c\alpha$$

where l_{xyz} is the total path length from i to o, and only single precision arithmetic is required.

- In a typical application, one knows the track position (x_i, y_i, z_i) and momentum at radius $r_i = \sqrt{x_i^2 + y_i^2}$ and one wants to propogate the track to the next measurement plane at radius r_j. If the B-field is inhomogenous, one must make short steps in arclength along a piecewise helical trajectory (see below) until one crosses the measuring plane; then back up. If the B-field is uniform, however, one can calculate the intersection point (x_j, y_j) of the track circle (in xy projection) and the circle of radius r_j:

$$(x_j - x_c)^2 + (y_j - y_c)^2 = r_c^2 \qquad \text{and} \qquad x_j^2 + y_j^2 = r_j^2$$

There are two solutions of the resulting quadratic equation for x_j (double precision is required):

$$r_0^2 x_j^2 - 2\delta^2 x_c x_j + (\delta^4 - r_j^2 y_c^2) = 0 \quad ; \qquad \delta^2 \equiv (1/2)(r_o^2 + r_j^2 - r_c^2) \quad ; \qquad y_j = \frac{\delta^2 - x_c x_j}{y_c}$$

One picks the solution which has

$$D = p_x(x_j - x_i) + p_y(y_j - y_i) > 0$$

and if both solutions have $D > 0$, pick the one corresponding to the smaller value of D. Now the turning angle from (x_i, y_i) to (x_j, y_j) can be calculated from the above, and the track can be swum to r_j.

Stereo Geometry

Stereo wires in a drift chamber are fixed at the endplates at the same value of r_i but are offset in ϕ. We define the *half*-length of the chamber to be L, the ϕ position of the wire at its midpoint ($z = 0$) to be ϕ_i, and its ϕ position at $z = \pm L$ to be $\phi_i \pm \Delta\phi_i$. The *stereo angle* is defined to be

$$\mu_i = \mathrm{atan}(r_i \sin \Delta\phi_i / L)$$

If the position resolution at layer i is σ_i, this corresponds to a resolution in z of $\sigma_z \approx \sigma_i/|\mu_i|$. Note that in the Mark II chamber, for example, the stereo wires have $\Delta\phi_i$ decreasing with r_i, to keep $|\mu_i|$ roughly constant for all stereo wires.

The radius as well as the ϕ position of the stereo wire depends on z:

$$\tan\phi(z) = \frac{(L-z)\sin(\phi_i - \Delta\phi_i) + (L+z)\sin(\phi_i + \Delta\phi_i)}{(L-z)\cos(\phi_i - \Delta\phi_i) + (L+z)\cos(\phi_i + \Delta\phi_i)}$$

$$R(z)^2/R^2 = 1 - (1 - z^2/L^2)\sin^2(\Delta\phi_i)$$

Of course, for small stereo angles, $\phi(z) \approx \phi_i + (z/l)\Delta\phi_i$, and $R(z) = R$.

Estimates of track parameter measurement error

We measure azimuthal position $d_i = (r\phi)_i$ at layer i, at radius r_i, with stereo angle μ_i. How does d_i depend on the track parameters? Use law of cosines on triangle of figure 1.

$$r_c^2 = (r_c + b)^2 + r_i^2 - 2r_i(r_c + b)\cos\beta$$

but $\cos\beta = \cos(\pi/2 - (\phi_i - \phi_0)) = \sin(\phi_i - \phi_0)$. Also, $\kappa = -q/(2r_c)$, so $4\kappa^2 = 1/r_c^2$.

$$\Rightarrow \qquad 1 = (1 + 2\kappa b)^2 + (2\kappa r_i)^2 - 4\kappa r_i(1 + 2\kappa b)\sin(\phi_i - \phi_0)$$

$$\Rightarrow \qquad \sin(\phi_i - \phi_0) = \left(\frac{\kappa}{1 + 2\kappa b}\right) r_i + \left(\frac{1 + \kappa b}{1 + 2\kappa b}\right)\left(\frac{b}{r_i}\right)$$

Define $\kappa' = \kappa/(1 + 2\kappa b)$ and $b' = b(1 + \kappa b)/(1 + 2\kappa b)$. Note that if $\kappa b \ll 1$ then $\kappa' \approx \kappa$ and $b' \approx b$. We have

$$\sin(\phi_i - \phi_0) = \kappa' r_i + b'/r_i$$

$$\Rightarrow \qquad \phi_i = \phi_0 + \mathrm{asin}(\kappa' r_i + b'/r_i)$$

$$\Rightarrow \qquad d_i = r_i\phi_0 + r_i \mathrm{asin}(\kappa' r_i + b'/r_i)$$

If the track is relatively stiff, and the impact parameter relatively small, we can replace the arcsine with its argument, getting the simple relation

$$d_i = b + r_i\phi_0 + r_i^2\kappa$$

To lowest order, $(\partial\phi/\partial r) = \kappa$. Including the arcsine (important for low momentum tracks), we get

$$\left(\frac{\partial\phi}{\partial r}\right)_i = \frac{\kappa'}{\sqrt{1-(\kappa' r_i)^2}} \qquad \Rightarrow \qquad \kappa' = \frac{(\partial\phi/\partial r)_i}{\sqrt{1+((\partial\phi/\partial r)_i r_i)^2}}$$

If the layer has a stereo angle μ_i, then the z-position will change the effective measured ϕ_i. We have

$$z_i = z_b + s\cdot l_i \qquad s \equiv p_z/p_t \qquad l_i \equiv r_c\alpha_i$$

where l_i is the path length in the xy plane, and α_i is the turning angle in the xy plane, as given above. This modifies the predicted d_i as given below.

Simple track fit. We temporarily assume that a track is stiff, and has a small impact parameter ($|\kappa|, |b| < 1; \kappa d \ll 1$). Then $l_i \approx r_i$. Define $\bar{\kappa} = \partial\phi/\partial r \approx \kappa$. Then we predict a track coordinate measurement at radius r_i:

$$d_i^0 = (b + r_i\phi_0 + r_i^2\bar{\kappa})\cos\mu_i + (z_b + r_i s)\sin\mu_i$$

Of course, if the stereo angle μ_i is small, we can approximate $\cos\mu_i = 1$ and $\sin\mu_i = \mu_i$, to get

$$d_i^0 = b + r_i\phi_0 + r_i^2\bar{\kappa} + \mu_i z_b + \mu_i r_i s$$

We now have a linear problem in the five track parameters $b, \phi_0, \bar{\kappa}, z_b, s$. We make n measurements of d_i^m, $i = 1, n$, at radii r_i, with stereo angle μ_i, and measurement error σ_i. Define $w_i = 1/\sigma_i^2$. Then

$$\chi^2 = \sum_{i=1}^{N} w_i\left[d_i - b - r_i\phi_0 - r_i^2\bar{\kappa} - \mu_i z_b - \mu_i r_i s\right]^2$$

$$\begin{pmatrix} \Sigma w_i & \Sigma w_i r_i & \Sigma w_i r_i^2 & \Sigma w_i\mu_i & \Sigma w_i\mu_i r_i \\ \Sigma w_i r_i & \Sigma w_i r_i^2 & \Sigma w_i r_i^3 & \Sigma w_i\mu_i r_i & \Sigma w_i\mu_i r_i^2 \\ \Sigma w_i r_i^2 & \Sigma w_i r_i^3 & \Sigma w_i r_i^4 & \Sigma w_i\mu_i r_i^2 & \Sigma w_i\mu_i r_i^3 \\ \Sigma w_i\mu_i & \Sigma w_i\mu_i r_i & \Sigma w_i\mu_i r_i^2 & \Sigma w_i\mu_i^2 & \Sigma w_i\mu_i^2 r_i \\ \Sigma w_i\mu_i r_i & \Sigma w_i\mu_i r_i^2 & \Sigma w_i\mu_i r_i^3 & \Sigma w_i\mu_i^2 r_i & \Sigma w_i\mu_i^2 r_i^2 \end{pmatrix} \begin{pmatrix} b \\ \phi_0 \\ \bar{\kappa} \\ z_b \\ s \end{pmatrix} = \begin{pmatrix} \Sigma w_i d_i \\ \Sigma w_i d_i r_i \\ \Sigma w_i d_i r_i^2 \\ \Sigma w_i d_i\mu_i \\ \Sigma w_i d_i\mu_i r_i \end{pmatrix}$$

The inverse of this matrix is an approximate version of a track parameter error matrix, assuming a relatively stiff track, with small impact parameter, passing through all layers. It is a quick way of estimating the track parameter resolution for a cylindrical tracker.

This matrix can be easily summed and inverted if the radii of the n measuring planes (each with spatial resolution σ_d) are evenly spaced from the origin to a maximum radius R; *i.e.* if $r_i = (i/n)R$; and if we use only axial layers to measure b, ϕ_0, κ and stereo layers to measure z_b, s. This gives the simple scaling rules for track parameter resolutions (for large n):

$$\sigma_b = \sqrt{9/n}\sigma_d$$

$$\sigma_{\phi 0} = \sqrt{\frac{192}{n}}\frac{\sigma_d}{R}$$

$$\sigma_\kappa = \sqrt{\frac{180}{n}}\frac{\sigma_d}{R^2} = \frac{0.3B}{2}\frac{\sigma_{pt}}{p_t^2}$$

$$\sigma_{zo} = \sqrt{4/n}(\sigma_d/\mu_i)$$

$$\sigma_s = \sqrt{\frac{12}{n}}\frac{\sigma_d/\mu_i}{R}$$

If the track exits through the endplate of a barrel tracker (at $z = \pm L$), then it has $|s| > s_{max} \equiv L/R$. We can estimate the degradation of the track parameter resolution by substituting $R \to L/|s|$ and $n \to n s_{max}/|s|$ in the above formulas.

Note that the resolution improves with the number of measurements like $1/\sqrt{n}$. But in the presence of multiple scattering, adding more measurements may *degrade* the resolution on some track parameters while improving it for others. In fitting for the impact parameters, for example, only the innermost precision measurements from the vertex tracker provide useful information; using all the hits with a least-squares estimator as described above is far from optimal. In the following sections we will discuss track fitting algorithms to optimize the track parameter resolution at the DCAO in the presence of multiple scattering.

<u>Forward trackers</u> have fixed z_i, not r_i. The imperfect knowledge of r_i does not, however, degrade the curvature measurement terribly. If $s = \tan\lambda$ of the track is known, one can replace $r_i = z_i/s$ in the above matrix trivially. In that case, one can analyze the problem with that simple substitution, and then the problem is linear in all track parameters except for s; the effect of forward detectors is to introduce larger correlations between s and the other track parameters; most importantly, the curvature measurement is now a function of s. Of course, the more general fit procedure described below (and in Trilling's memo) handles forward trackers as well as barrel trackers.

Piecewise-helical track trajectory

If the B-field is not a uniform axial field, the track can be propogated as a set of linked helical segments, each of which is a true orbit for a constant field equal to the actual field near the center of the segment (*cf.* Trilling). This "track swim" comes in handy when doing a track fit, or when generating track trajectories in a Monte Carlo.

At any point on the trajectory, we have the track position vector $\vec{P} = (x_i, y_i, z_i)$ and unit vector $\hat{t}$ tangent to the momentum vector at $\vec{P}$. We want to find a new position $\vec{P'}$ and unit vector $\hat{t'}$ at a distance Δl (in xyz) along the trajectory. We define the unit vector in the direction of the B-field $\hat{h}_3$; the unit vector perpindicular to $\hat{h}_3$ and $\hat{t}$:

$$\hat{h}_1 = \frac{\hat{t} \times \hat{h}_3}{|\hat{t} \times \hat{h}_3|};$$

and $\hat{h}_2 = \hat{h}_3 \times \hat{h}_1$, which is tangent to $\hat{t}$ in the plane perpindicular to the B-field. If the B-field is in the $\hat{z}$ direction, then $\hat{h}_1$ and $\hat{h}_2$ are in the xy plane. The unit vectors $\hat{h}_1, \hat{h}_2, \hat{h}_3$ form an orthonormal right-handed coordinate system. We define the local dip angle:

$$\cos\lambda = \hat{t} \cdot \hat{h}_2, \qquad \sin\lambda = \hat{t} \cdot \hat{h}_3$$

The local radius of curvature is $r_c = p\cos\lambda/0.3B$, B in Tesla, p in GeV/c. The turning angle in the plane perpindicular to B-field is $\alpha = \Delta l \cos\lambda/r_c$, where the xyz path length Δl is in meters. Then, after a step Δl, the new ("swum") position and tangent vectors are

$$\vec{P'} = \vec{P} + r_c[q(1-\cos\alpha)\hat{h}_1 + \sin\alpha\hat{h}_2 + +\alpha\tan\lambda\hat{h}_3]$$

$$\begin{aligned} \hat{t'} &= (\hat{t'} \cdot \hat{h}_1)\hat{h}_1 + (\hat{t'} \cdot \hat{h}_2)\hat{h}_2 + (\hat{t'} \cdot \hat{h}_3)\hat{h}_3 \\ &= q\sin\alpha\cos\lambda\hat{h}_1 + \cos\alpha\cos\lambda\hat{h}_2 + \sin\lambda\hat{h}_3 \end{aligned}$$

<u>Including multiple scattering</u> At any point on the track trajectory, one can include the effect of multiple scattering in a plane of material. There are two angles, ϕ_k and θ_k, at every plane k. In a Monte Carlo, one throws each of these angles as Gaussians of width

$$\sigma_\phi = \sigma_\theta = \frac{0.0141}{p\beta}\sqrt{\frac{l_{xyz}}{X_0}}$$

where l_{xyz} is the total path length of the track through the scattering material, of radiation length X_0.

One then defines two unit vectors perpindicular to $\hat{t}$; for example,

$$\hat{\phi} = \frac{\hat{t} \times \hat{z}}{|\hat{t} \times \hat{z}|}; \qquad \hat{\theta} = \hat{t} \times \hat{\phi}$$

After a small scatter, the new momentum direction $\hat{t}'$ is

$$\hat{t}' = \frac{\hat{t} + \phi_k \hat{\phi} + \theta_k \hat{\theta}}{\sqrt{1 + \phi_k^2 + \theta_k^2}}$$

where we have assumed that $\phi_k, \theta_k \ll 1$.

The Track Error Matrix

Let the five (or more) track parameters be P_α, $\alpha = 1$ to 5 ($P_\alpha = b, \phi_0, \kappa, z_b, s$). Call d_i, $i = 1$ to n the distance of closest approach of the track to a wire (DCAW) at layer i as calculated from the track parameters (see section on piecewise helical track trajectory, above), while d_i^m are the measured DCAW's. We approximate

$$d_i \approx d_i^0 + \frac{\partial d_i}{\partial P_\alpha} \Delta P_\alpha.$$

Approximate formulas for the derivatives are easy to obtain from the above:

$$\partial d_i / \partial b = \cos \mu_i \qquad \partial d_i / \partial \phi_0 = r_i \cos \mu_i \qquad \partial d_i / \partial \kappa = r_i^2 \cos \mu_i$$

$$\partial d_i / \partial z_b = \sin \mu_i \qquad \partial d_i / \partial s = r_i \sin \mu_i$$

Defining the spacelike residual $r_i = d_i^m - d_i$, the track fit is the minimum of the fit χ^2:

$$\chi^2 = \sum_i \frac{1}{\sigma_i^2} r_i^2 = \sum_i w_i \left(d_i^m - d_i^0 - \frac{\partial d_i}{\partial P_\alpha} \Delta P_\alpha \right)^2 .$$

We usually take the weights w_i from the observed residual distributions. The solution is:

$$\Delta P_\alpha = \left[\sum_i w_i \frac{\partial d_i}{\partial P_\alpha} \frac{\partial d_i}{\partial P_\beta} \right]^{-1} \left[\sum_i w_i (d_i^m - d_i^0) \frac{\partial d_i}{\partial P_\beta} \right] \equiv M_{\alpha\beta}^{-1} V_\beta$$

where the definitions of $M_{\alpha\beta}$ (the *track parameter inverse error matrix*) and V_β are obvious. The track fit consists of calculating the d_i^0 and $\frac{\partial d_i}{\partial P_\alpha}$ from initial track parameters P_α^0, calculating $M_{\alpha\beta}^{-1}$ (inverting a 5×5 matrix) and V_β, improving one's values for the $P_\alpha = P_\alpha^0 + \Delta P_\alpha$, and iterating (because of the linear approximation for d_i). In the Mark II drift chamber, one rarely needs more than two or three iterations.

We can now take the variance with respect to measurement errors, square, and average, giving the expected track parameter error matrix:

$$\sigma^2_{\alpha\beta} \equiv \langle \delta P_\alpha \delta P_\beta \rangle = M^{-1}_{\alpha\gamma} M^{-1}_{\beta\rho} \left[\sum_{i,j} w_i w_j \frac{\partial d_i}{\partial P_\gamma} \frac{\partial d_j}{\partial P_\rho} \langle \delta d_i^m \delta d_j^m \rangle \right] .$$

Now we take

$$\langle \delta d_i^m \delta d_j^m \rangle = \sigma_i^2 \delta_{ij} + \sigma_{ij}^2$$

where σ_i is the rms of the uncorrelated measurement errors, while σ_{ij} gives the correlations between measurement errors from hits in different layers (due, *e.g.*, to geometrical or timelike correlations; for example, putting more than one wire in the same feedthrough or the same TDC module).

Carrying along correlated as well as uncorrelated errors, we get:

$$\sigma^2_{\alpha\beta} = (w_i \sigma_i^2) M^{-1}_{\alpha\beta} + M^{-1}_{\alpha\gamma} M^{-1}_{\beta\rho} \left[\sum_{i,j} w_i w_j \sigma_{ij}^2 \frac{\partial d_i}{\partial P_\gamma} \frac{\partial d_j}{\partial P_\rho} \right] .$$

If we are careful to choose $w_i = 1/\sigma_i^2$ for all layers, then in the absence of correlations, the track parameter error matrix is simply $M^{-1}_{\alpha\beta}$.

The correct way to handle correlations if $\left\langle \delta d_i^m \delta d_j^m \right\rangle \equiv W^{-1}_{ij}$ is not diagonal, is to define the χ^2 to be

$$\chi^2 = \sum_{i,j} (d_i^m - d_i) W_{ij} (d_j^m - d_j)$$

We could then define

$$M_{\alpha\beta} = \sum_{ij} \frac{\partial d_i}{\partial P_\alpha} W_{ij} \frac{\partial d_j}{\partial P_\beta} \qquad \text{and} \qquad V_\beta = \sum_{ij} (d_i^m - d_i^0) W_{ij} \frac{\partial d_j}{\partial P_\beta}$$

and then the track parameter error matrix is $M^{-1}_{\alpha\beta}$ even in the presence of correlations. Of course, if there are n hits, accumulating these quantities takes $(n-1)/2$ times as long. As will be shown below, correlations invariably serve to degrade the track parameter resolution.

<u>Multiple scattering errors</u> One simple way to include multiple scattering in the track parameter error matrix is to choose w_i in the final fit to include the effect of multiple scattering at layer i, based on a preliminary estimate of the track momentum. This ignores correlations between measurements, however.

We can include the correlations by proceeding as follows. With m scattering planes at radii $r_k, k = 1$ to m, the added contribution to the measurement error δd_i^m is just

$$\delta d_i^m \to \delta d_i^m + \sum_{k<i} \phi_k \cos \mu_i (r_i - r_k) \text{sec}\lambda + \sum_{k<i} \theta_k \sin \mu_i (r_i - r_k) \text{sec}^2 \lambda,$$

where ϕ_k is the xy multiple scattering angle and θ is the rz multiple scattering angle at scattering plane k; and $\text{sec}\lambda = p/p_t$. It is clear that multiple scattering introduces unavoidable random correlations between all hits in a track.

This will contribute to the track error matrix by adding a term that looks like

$$\begin{aligned} \left\langle \delta d_i^m \delta d_j^m \right\rangle = \sigma_i^2 \delta_{ij} + \sec^2 \lambda \sum_{k<i,j} \left\langle \phi_k^2 \right\rangle \cos \mu_i \cos \mu_j (r_i - r_k)(r_j - r_k) \\ + \sec^4 \lambda \sum_{k<i,j} \left\langle \theta_k^2 \right\rangle \sin \mu_i \sin \mu_j (r_i - r_k)(r_j - r_k) \end{aligned}$$

The track error matrix then becomes

$$\begin{aligned} \sigma_{\alpha\beta}^2 =& M_{\alpha\beta}^{-1} + \sec^2 \lambda M_{\alpha\gamma}^{-1} M_{\beta\rho}^{-1} \left[\sum_{i,j} \frac{1}{\sigma_i^2 \sigma_j^2} \frac{\partial d_i}{\partial P_\gamma} \frac{\partial d_j}{\partial P_\rho} \sum_{k<i,j} \left\langle \phi_k^2 \right\rangle \cos \mu_i \cos \mu_j (r_i - r_k)(r_j - r_k) \right] \\ &+ \sec^4 \lambda M_{\alpha\gamma}^{-1} M_{\beta\rho}^{-1} \left[\sum_{i,j} \frac{1}{\sigma_i^2 \sigma_j^2} \frac{\partial d_i}{\partial P_\gamma} \frac{\partial d_j}{\partial P_\rho} \sum_{k<i,j} \left\langle \theta_k^2 \right\rangle \sin \mu_i \sin \mu_j (r_i - r_k)(r_j - r_k) \right] \end{aligned}$$

We can then take:

$$\left\langle \phi_k^2 \right\rangle = \left\langle \theta_k^2 \right\rangle = \left(\frac{0.0141}{p\beta} \right)^2 \frac{\Delta L}{X_0}$$

where p is the momentum in GeV/c, β is the track velocity, ΔL is the track length through the scattering plane ($= \Delta R \text{sec}\lambda$ for cylindrical scattering plances of radial thickness ΔR), and X_0 is the radiation length of the scattering material.

These sums can be done numerically for a given detector and track. Here is how they scale, for a detector of radial extent R, with distributed material (gas and wires) of radiation length X_0. We note that matrix elements generally scale like

$$M_{\alpha\beta}^{-1} \sim \frac{\sigma^2}{\frac{\partial d_i}{\partial P_\alpha} \frac{\partial d_i}{\partial P_\beta}}$$

But $\frac{\partial d_k}{\partial \phi_0} \sim r_i$, and so on, so:

$$\langle \delta\phi_0 \delta\phi_0 \rangle \approx 0.23 \left(\frac{0.0141}{p\beta} \right)^2 \frac{R}{X_0} \sec^3 \lambda$$

$$\langle \delta\phi_0 \delta\kappa \rangle \approx \frac{0.21}{2} \left(\frac{0.0141}{p\beta} \right)^2 \frac{1}{X_0} \sec^3 \lambda$$

$$\langle \delta\kappa \delta\kappa \rangle \approx \frac{1.43}{4} \left(\frac{0.0141}{p\beta} \right)^2 \frac{1}{RX_0} \sec^3 \lambda$$

$$\langle \delta s \delta s \rangle \approx 0.20 \left(\frac{0.0141}{p\beta} \right)^2 \frac{R}{X_0} \sec^5 \lambda$$

This allows one to include the effects of multiple scattering into the track parameter error matrix.

Fits including scattering angle. Alternatively, we can include the scattering angles ϕ_k, $k = 1, m$ as track parameters P_{5+k} (remembering that there are two of them at every scattering plane). In that case, one can improve the track parameter estimates themselves, not just the error matrix. One need only have expressions for $\frac{\partial d_i}{\partial P_\alpha}$ for each of the measurement plane/scattering plane combinations; these are simply $(r_i - r_k)$, for small scatters and relatively stiff tracks (more precisely, $\frac{\partial d_i}{\partial P_\alpha} = (\hat{d} \cdot \hat{\phi})(r_i - r_k)$, where $\alpha = 5 + k$, $\hat{d}$ is the unit vector in the direction of the measurement, *e.g.*, the azimuthal direction for axial wires, and $\hat{\phi}$ is the scattering angle direction; see piecewise-helical trajectory discussion, above). One then applies the constraint that the scattering angle be consistent with the expected Gaussian error (calculated from the amount of traversed material), by adding to the fit χ^2 a term of the form

$$\chi^2 = \sum_i w_i (d_i^m - d_i^0 - \frac{\partial d_i}{\partial P_\alpha} \Delta P_\alpha)^2 + \sum_k \frac{\phi_k^2}{\langle \phi_k^2 \rangle}$$

We can write this in a more general way by defining $w_\alpha = 1/\langle \phi_k^2 \rangle$ for $\alpha = 5 + k$, and 0 otherwise. We then note that

$$\phi_k = \Delta\phi_k + \phi_k^0 = \Delta P_\alpha + P_\alpha^0$$

for $\alpha = 5 + k$, where ϕ_k^0 is the value of the scattering angle from the previous iteration of the fit. The χ^2 is then written

$$\chi^2 = \sum_i w_i (d_i^m - d_i^0 - \frac{\partial d_i}{\partial P_\alpha} \Delta P_\alpha)^2 + \sum_\alpha w_\alpha (\Delta P_\alpha + P_\alpha^0)^2$$

The χ^2 is minimized, yielding (no summation convention):

$$\sum_\beta \Delta P_\beta (M_{\alpha\beta} + w_\alpha \delta_{\alpha\beta}) = V_\alpha - w_\alpha P_\alpha^0$$

We can redefine the $(5 + m) \times (5 + m)$ inverse error matrix, and the $(5 + m)$ measurement vector to be

$$M_{\alpha\beta} \rightarrow \left[\sum_i w_i \frac{\partial d_i}{\partial P_\alpha} \frac{\partial d_i}{\partial P_\beta} \right] + w_\alpha \delta_{\alpha\beta}$$

$$V_\beta \rightarrow \left[\sum_i w_i (d_i^m - d_i^0) \frac{\partial d_i}{\partial P_\beta} \right] - w_\beta P_\beta^0$$

We then fit for $\Delta P_\alpha = M_{\alpha\beta}^{-1} V_\beta$ as above. One must invert a $(5+m) \times (5+m)$ matrix at every iteration of the track fit.

When evaluating the error matrix, we get

$$\sigma_{\alpha\beta}^2 \equiv \left\langle \delta P_\alpha \delta P_\beta \right\rangle = M_{\alpha\gamma}^{-1} M_{\beta\rho}^{-1} \left\{ \left[\sum_{i,j} w_i w_j \left\langle \delta d_i^m \delta d_j^m \right\rangle \frac{\partial d_i}{\partial P_\gamma} \frac{\partial d_j}{\partial P_\rho} \right] + w_\gamma w_\rho \left\langle \delta P_\gamma^0 \delta P_\rho^0 \right\rangle \right\}$$

If the weights are chosen well, then $\left\langle \delta d_i^m \delta d_j^m \right\rangle = \delta_{ij}/w_j$ (or W_{ij} if the measurement errors are correlated) and $\left\langle \delta P_\gamma^0 \delta P_\rho^0 \right\rangle = \delta_{\gamma\rho}/w_\rho$ (there are no correlations amongst the scattering angles ϕ_k). Then $\left\langle \delta P_\alpha \delta P_\beta \right\rangle = M_{\alpha\beta}^{-1}$. The net effect is a better measurement of the track parameters P_α, $\alpha = 1,5$ evaluated at the DCAO, and smaller expected errors on these quantities. (This method was employed by Trilling in the Mark II many years before it was "invented" by, *e.g.*, Lutz (1987)).

In this method, one can easily end up with more fit parameters than measurements. In addition, one can end up inverting large matrices many times, which is time-consuming. A common technique to deal with these problems is to reduce the number of fitted scattering angles by neglecting scattering in the rz plane, and lumping the scattering into planes which lie in between different detector subsystems and/or superlayers.

Yet another method, due to Billoir (NIM 225 (1984) 352), is a novel way to obtain an optimal measure of the five track parameters and their errors at the DCAO. The method is to work backwards from the end of the track trajectory, swimming the track parameters and the error matrix, both evaluated at measurement location i, back to measurement location $i-1$; one refits for the track parameters with the information from measurement $i-1$ and inflates the appropriate elements of the error matrix to account for the (unknown) scattering there. Iterating this procedure, eventually one adds all points to the fit and arrives at the DCAO, yielding the optimal estimate for the track parameters there, and their error matrix. This method requires inverting a 5×5 matrix at every measurement plane.

<u>Effect of correlated errors on track parameter errors</u> Drift chambers such as the Mark II and SLD have more than one wire in a jet cell; a track passing though the cell leaves hits in all wires of the cell. The wires in a cell are strung on one feedthrough, and often go into a

common set of electronics; so they have correlated geometrical and timelike errors. Now we make the approximation

$$\frac{\partial d_i}{\partial P_\gamma} \approx \frac{\partial d_j}{\partial P_\gamma}$$

for i, j in the same cell. If there are m wires in a cell ($m = 6$ for Mark II and 8 for SLD), then $\sigma_{ij} = m\sigma_{corr}^2 \delta_{ij}$, and we get

$$\sigma_{\alpha\beta}^2 = w_i \sigma_i^2 M_{\alpha\beta}^{-1} + m w_i \sigma_{corr}^2 M_{\alpha\beta}^{-1} = M_{\alpha\beta}^{-1} \frac{\sigma_i^2 + m\sigma_{corr}^2}{\sigma_i^2 + \sigma_{corr}^2},$$

where in the last step we have taken w_i from residual distributions; $1/w_i = \sigma_i^2 + \sigma_{corr}^2$. This shows that correlated errors in a cell will be magnified by a factor of m relative to uncorrelated errors, yielding poorer resolution in the track parameters than would be expected from the residual distributions (in the Mark II drift chamber, this is a $\sim 40\%$ effect!). We see that the jet cell geometry of a drift chamber, which makes pattern recognition so much easier, can have serious effects on the track parameter resolution.

Track Fitting

The enclosed memo by Trilling describes the use of this formalism in doing track fits. Here I only outline the steps:

1. Using initial track fit parameters (P_α^0), propogate the track trajectory to each of the measurement layers. The track parameters may include one or more scattering angles ϕ_k at specific radii k.
2. At each measurement layer, determine the predicted measurement d_i^0 (*e.g.*, drift distance or silicon strip coordinate), as well as the expected weight for that quantity, $w_i = 1/\sigma_i^2$. Calculate the approximate linear derivatives of the measurement with respect to the track parameters, $\frac{\partial d_i}{\partial P_\alpha}$. Accumulate $M_{\alpha\beta}$ and V_β. Accumulate the fit χ^2, including multiple scattering angles if these are in the fit:

$$\chi^2 = \sum_i \frac{(d_i^m - d_i)^2}{\sigma_i^2} + \sum_k \frac{\phi_k^2}{\langle \phi_k^2 \rangle} = \sum_i w_i (d_i - d_i^0)^2 + \sum_\alpha w_\alpha P_\alpha^2$$

3. Invert $M_{\alpha\beta}$ and solve for ΔP_α. Iterate until the track parameters are stable. The last $M_{\alpha\beta}$ is the inverse of the error matrix for the P_α.

Pattern recognition by segment finding

If the layers in a tracker can be grouped into logical superlayers, close together in space, with similar orientation and resolution, one can do local pattern recognition within that superlayer, reconstructing track segments (or vectored hits). When associating hits, one gives special attention to the following:

- For a drift chamber jet cell, one resolves the left-right ambiguity locally, paying attention to cases where tracks cross one another and/or cell boundaries or the sense wire plane. Lorentz angle effects and double–track separation must be taken into account when making cuts.
- Local "iterative finishing": a second pass where one picks up good hits and throws out bad hits, based on a fast straight-line fit χ^2.

Segment track parameters. The local straight-line fit yields a measure of the track parameters for a track passing through the origin and the segment, and the correlated errors for those parameters. Define $\bar{\kappa} = \partial\phi/\partial r$. Then

$$\begin{pmatrix} \Sigma w_i r_i^2 & \Sigma w_i r_i^3 \\ \Sigma w_i r_i^3 & \Sigma w_i r_i^4 \end{pmatrix} \begin{pmatrix} \phi_0 \\ \bar{\kappa} \end{pmatrix} = \begin{pmatrix} \Sigma w_i d_i r_i \\ \Sigma w_i d_i r_i^2 \end{pmatrix}$$

One can also calculate $\bar{r} = \Sigma r_i/N$ and $\bar{\phi} = (\Sigma d_i/r_i)/N$, where N is the number of hits in the segment. Then one can improve on the measurements of κ and ϕ_0 by taking into account the arcsine in the equation for ϕ_i:

$$\kappa = \bar{\kappa}/\sqrt{1+(\bar{\kappa}\bar{r})^2}; \qquad \phi_0 = \bar{\phi} - \mathrm{asin}(\kappa\bar{r})$$

The above 2×2 matrix is the inverse of the error matrix for the measurement of the two track parameters. Segments from the same track will lie nearby in κ, ϕ_0 space, and this matrix is the "metric" in that 2D space for establishing that two segments are consistent with coming from the same track (see segment association below).

If the segment consists of only two measurements,

$$\bar{\kappa} = \frac{(r\phi)_2/r_2 - (r\phi)_1/r_1}{r_2 - r_1}; \qquad \bar{r} = \frac{r_1 + r_2}{2}; \qquad \bar{\phi} = \frac{(r\phi)_1/r_1 + (r\phi)_2/r_2}{2}$$

and the above two equations yield ϕ_0 and κ.

If the superlayer has stereo wires, all of which have stereo angle μ, then one is measuring $\kappa + s\mu$ rather than κ, and $\phi_0 + z_b\mu$ rather than ϕ_0. Segments from superlayers with the same value of μ can be clustered. If a stereo segment is associated with a track which has been found in xy projection, then the above measurements yield measurements of z_b and s. Alternatively, one can convert the ϕ_i measurements into measurements of z_i (with resolution σ/μ) and fit for z_b, s directly.

Superlayers in forward detectors (fixed in z_i, not r_i) can also be found by matching hits that form straight-line segments (an origin constraint can be used if the superlayer consists of only two layers). They too measure track parameters. In silicon microstrip or radial drift chambers that measure ϕ_i, with r_i measured by small-angle stereo, we have:

$$\phi_i = \phi_0 + \mathrm{asin}(\kappa r_i); \qquad \text{but} \quad z_i = z_b + (s/\kappa)\mathrm{asin}(\kappa r_i)$$

$$\Rightarrow \qquad \phi_i = \phi_0 + (\kappa/s)(z_i - z_b) = (\phi_0 - \kappa z_b/s) + (\kappa/s)z_i$$

So we can measure $\partial\phi_i/\partial z_i = \kappa/s$, with intercept $(\phi_0 - \kappa z_b/s)$. Small angle stereo in silicon microstrips or charge division in radial drift chambers gives

$$r_i = \frac{1}{\kappa}\sin\left(\frac{\kappa}{s}(z_i - z_b)\right)$$

The sine can easily be approximated by its argument if the resolution on r_i is not great. We measure $\partial r_i/\partial z_i = 1/s$, with intercept $-z_b/s$. Again, we get a 2×2 matrix to cluster segments in forward detectors. Note that the quantities which are measured with Gaussian resolution in the barrel detector are different than the ones so measured in the forward detectors; so clustering segments from both types of detectors is difficult.

Vectored hit finding in silicon. Vectored hit finding in a two layer superlayer is simplicity: match hits with $|\phi_2 - \phi_1|$ minimized (loop over all ϕ_1 in layer 1, find closest ϕ_2 within the detector). This is a bias towards high momentum tracks. It works especially well for vertex chambers (but not for tracks with large impact parameter).

Stereo matching is also straightforward. Tracks from the origin ($z_b = 0$) have constant $s = z_2/r_2 = z_1/r_1$. So once we have a ϕ pair, we choose the stereo assignment that minimizes $|r_2 z_1 - r_1 z_2|$. For double-sided silicon microstrips with small-angle stereo on one side, we can do local pattern recognition, finding the vectored hit in 3D.

In a silicon strip detector, strip number can be calculated from the ϕ_i, z_i of the track at that layer:

$$IS_i = Int\left[(r_i\phi_i - \mu_i z_i)/w + 1\right]$$

where w is the strip width; and axial strips have $\mu_i = 0$. Given axial strip IS_i, we can reconstruct ϕ_i:

$$\phi_i \approx (IS_i - \frac{1}{2}) \cdot w/r_i$$

with a "box" resolution function of width w, $\sigma = w/\sqrt{12}$.

A match between an axial strip IS_i and a stereo strip JS_i at the same radius r_i (in other words, double–sided strips), yields

$$z_i \approx [IS_i - JS_i] \cdot w/\mu_i$$

with a "triangle" resolution function, $\sigma = w/(\sqrt{6}\mu_i)$. In these relations, care must be taken to handle the cases when the effective ϕ of the hit crosses the $2\pi \to 0$ boundary.

Forming segments is easy. Match hits in layer i with those in layer j. For a given hit ϕ_i in layer i, match with the hit in layer j which minimizes $|\phi_i - \phi_j|$. Accept only those segments with $|\phi_i - \phi_j| \le |r_i - r_j|\kappa_{max}$, where κ_{max} is the curvature associated with the minimum p_t track of interest.

The stereo hits ϕ_{si} and ϕ_{sj} give the ϕ of the strip at $z = 0$. We then have

$$\phi_{si} \approx \phi_i + \mu_i z_i / r_i \approx \phi_i + \mu_i(z_b + sr_i)/r_i \approx \phi_i + s\mu_i$$

If we arrange our superlayers such that $\mu_i = -\mu_j$, then we simply choose the ϕ_{si}, ϕ_{sj} pair that best satisfies

$$\phi_{si} + \phi_{sj} = \phi_i + \phi_j$$

There are simple scaling rules which allow one to estimate the efficacy of these pattern recognition algorithms, given the tracker resolution and geometry, and the jettiness of the physics of interest. See Sadrozinski *et al.*, NIM A279 (1989) 223.

Segment Association and Clustering

A single track will leave segments in each of the superlayers it passes through. If those segments are found correctly, they will cluster together in the space of track parameters; one hopes as well that they separate away from clusters of segments from other tracks, even in projected subspaces of the total track parameter space. It is the job of the tracker designer to ensure that the resolution and geometry of the tracker is a match for the multiplicity and jettiness of the target physics.

A *cluster* is a set of all reconstructed track segments in an event which *associate* with at least one other segment in the cluster, and which is irreducible - can't be split into two or more clusters whose members don't associate with vectors in another cluster.

Segments *associate* with each other if they lie "nearby" each other in the space of track parameters (or some projected subspace), measured by some criterion. A good measure is a χ^2 test, which works with two or more segments:

$$\vec{V}_i \equiv \begin{pmatrix} \phi_{0i} \\ \kappa_i \end{pmatrix}; \qquad Minv_i \equiv \begin{pmatrix} \sigma^2_{\phi_0\phi_0} & \sigma^2_{\phi_0\kappa} \\ \sigma^2_{\phi_0\kappa} & \sigma^2_{\kappa\kappa} \end{pmatrix}$$

$$\chi^2 = \sum_i (\vec{V}_i - \vec{V}_0) M_i (\vec{V}_i - \vec{V}_0)$$

$$\Rightarrow \qquad \vec{V}_0 = \left(\sum_j M_j\right)^{-1} \left(\sum_i M_i \vec{V}_i\right)$$

from which one easily calculates the minimized χ^2, choosing an appropriate cut which maximizes the efficiency for clustering segments from the same track while minimizing the chance of clustering segments from nearby tracks. This test can be employed in any 2×2 space, not just (ϕ_0, κ). It is also easily generalized to $n \times n$ spaces of more track parameters; but see the comment at the end of this section for an important caution!

In the presence of multiple scattering, one can use the local estimate of κ (and secλ, if stereo information is available locally) to modify the 2×2 error matrix $Minv_i$ appropriately. Using the approximate forms given in section 5 for the contribution of multiple scattering to the track parameter error matrix, we have, for a segment at radius R,

$$M_i^{-1} = \begin{pmatrix} \Sigma w_i r_i^2 & \Sigma w_i r_i^3 \\ \Sigma w_i r_i^3 & \Sigma w_i r_i^4 \end{pmatrix}^{-1} + \left(\frac{0.0141}{p_t \beta}\right)^2 \frac{\sec\lambda}{X_0} \begin{pmatrix} 0.23R & 0.21(2/0.3B) \\ 0.21(2/0.3B) & 1.43/R \cdot (2/0.3B)^2 \end{pmatrix}$$

One must invert the first 2×2 matrix on the rhs, add the second, and then invert again to get M_i and form the segment association χ^2.

Arbitrating clusters is easy for isolated tracks. If the cluster contains no more than one segment per superlayer, then it is only one track; pattern recognition is done. Nearby tracks, however, may merge to form one cluster; then more than one segment from the same superlayer will appear in the cluster. Several techniques exist to arbitrate such *complex clusters*:

- If the segments separate into two or more clusters except for one ambiguous superlayer (whose segment(s) associate with both subclusters) one can eliminate that one layer, then arbitrate the ambiguous segments(s) using more segments in the above χ^2.
- Failing this, one looks for the least ambiguous superlayer in that cluster; this is the superlayer with the maximum number of other superlayers unambiguously associated with it. One build unambiguous subclusters around that superlayer, arbitrating segments in ambiguous superlayers between the unambiguous subclusters using a best-χ^2 test.
- Failing this, one can laboriously loop through all combinations of ambiguous superlayers, choosing the combination with the best χ^2; since only 2×2 matrix operations are required for each combination, this can work fast.
- If none of this works, chances are you have reached the resolving power of your detector; you can tell how many tracks are in the complex cluster, but you can't separate them.

One must take care in forming clusters. One clusters "loosely" to get a sufficient number of segments clustered, in order to consider a track found. Lost segments can be picked up, and bad ones dropped, at an "iterative finishing" stage. But if one clusters too loosely, tracks that may have formed distinct clusters will merge into one cluster; pattern recognition becomes much more difficult. A general rule of thumb is that adding more information (such as stereo coordinate information or information from a less precise chamber) must serve only to *tighten* rather than *loosen* the clustering; otherwise, tracks that may have separated well in (ϕ_0, κ) space may merge together when (z_b, s) information is added; one thinks one is gaining efficiency, but instead one only makes pattern recognition more difficult.