First order radiative corrections to polarized muon decay spectrum

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Abstract

The first order QED corrections to the polarized muon decay spectrum are considered. The exact dependence on electron and muon masses is kept. Numerical results are presented.

Key words: muon decay, radiative corrections

1 Introduction

Since the discovery of muon in 1936, experimental and theoretical investigations of its properties became an important part of the elementary particle physics. The very accurate measurements of the properties provide serious checks of the Standard Model and give a possibility to look for new physics in low–energy experiments. Because of the more and more precise experimental facilities and techniques, calculations of radiative corrections become unavoidable. We need them to obtain theoretical predictions with the required accuracy.

In this paper we will consider radiative corrections to the polarized muon decay spectrum. This work was initialized by the experiment TWIST [1,2], which is going to measure the spectrum with the error level of the order of 10^{-4}. We will speak here only about the first order quantum electrodynamic (QED)

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correction and concentrate on the effect of the non–zero ratio of the electron and the muon masses. Higher order corrections to the muon decay spectrum will be considered elsewhere [3,4]. A review about the muon properties and possible non–standard effects in the muon decay can be found in Refs. [5,6].

2 The Born level

At the Born level within the Fermi Model the differential width is described by the formula [7,8]:

$$
\begin{aligned}
&\frac{\mathrm{d}^2\Gamma^{\mathrm{Born}}}{\mathrm{d}x\mathrm{d}c} = \Gamma_0 x^2\beta\left(f^{\mathrm{Born}}(x) + c\xi g^{\mathrm{Born}}(x)\right), \qquad \Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3}, \\
&f^{\mathrm{Born}}(x) = 3 - 2x + \frac{x}{4}(3x-4)(1-\beta^2), \\
&g^{\mathrm{Born}}(x) = (1-2x)\beta + \frac{3x^2}{4}(1-\beta^2)\beta, \\
&\qquad \beta = \sqrt{1-\frac{m_e^2}{E_e^2}}, \qquad E_e = \frac{m_\mu}{2}x, \qquad c = \cos\theta,
\end{aligned} \tag{1}
$$

where m_e and m_μ are the electron and the muon masses; θ is the angle between the electron momentum and the muon polarization vector ($c \to -c$ for the μ^+ decay); ξ is the degree of the muon polarization; β is the electron velocity in the muon rest reference frame; E_e is the electron energy; x is the electron energy fraction,

$$
x_{\mathrm{min}} < x < x_{\mathrm{max}}, \qquad x_{\mathrm{min}} = 2\sqrt{\rho}, \qquad x_{\mathrm{max}} = 1+\rho, \qquad \rho = \frac{m_e^2}{m_\mu^2}. \tag{2}
$$

In the massless limit ($m_e \to 0$) we have

$$
f^{\mathrm{Born}}(x) \to f_0(x) = 3 - 2x, \qquad g^{\mathrm{Born}}(x) \to g_0(x) = 1 - 2x. \tag{3}
$$

The integration over the energy fraction gives

$$
\begin{aligned}
&\int\limits_{x_{\mathrm{min}}}^{x_{\mathrm{max}}}\mathrm{d}x\; x^2\beta f^{\mathrm{Born}}(x) = \frac{1}{2}F(\rho), \qquad \int\limits_{x_{\mathrm{min}}}^{x_{\mathrm{max}}}\mathrm{d}x\; x^2\beta g^{\mathrm{Born}}(x) = \frac{1}{2}G(\rho), \\
&F(\rho) = 1 - 8\rho - 12\rho^2\ln\rho + 8\rho^3 - \rho^4, \\
&G(\rho) = -\frac{1}{3} + \frac{32}{3}\rho^{3/2} - 30\rho^2 + 32\rho^{5/2} - \frac{40}{3}\rho^3 + \rho^4.
\end{aligned} \tag{4}
$$

Function $F(\rho)$ is relevant for the total decay width at the Born level:

$$\Gamma^{\mathrm{Born}} = \int\limits_{-1}^{1} \mathrm{d}c\ \Gamma_0 \frac{F(\rho)}{2} = \Gamma_0 F(\rho). \tag{5}$$

Function $G(\rho)$ contributes to the forward–backward asymmetry of the decay:

$$\Gamma^{\mathrm{Born}}_{\mathrm{FB}} = \int\limits_{0}^{1} \mathrm{d}c\ c\xi\ \Gamma_0 \frac{G(\rho)}{2} - \int\limits_{-1}^{0} \mathrm{d}c\ c\xi\ \Gamma_0 \frac{G(\rho)}{2} = \xi \Gamma_0 G(\rho). \tag{6}$$

Within the Standard Model the muon decay happens due to weak interaction of leptons and W-bosons. The Fermi Model corresponds to the limiting case of the infinite W-boson mass. We follow here the definition of the Fermi coupling constant G_F as discussed in Ref. [9]. That means, all weak effects are incorporated into the coupling constant, and QED radiative corrections have to be calculated within the Fermi Model. I accept and support this approach. Originally in the literature, the constant is defined in a different way [10–12], so that the first order effect in the muon and the W-boson mass ratio gives

$$\Gamma_0 \longrightarrow \Gamma_0 \biggl(1 + \frac{3}{5}\,\frac{m_\mu^2}{m_W^2} \biggr). \tag{7}$$

But in any case, in studies of the muon decay spectrum, it is natural to use the constant directly defined from very precise experiments on the muon lifetime. We have to note here, that although the Fermi lagrangian itself is not renormalizable, QED corrections to the process under consideration can be shown to be finite [13] at all orders of the perturbation theory.

3 The Exact First Order QED Corrections

Here we will consider the $\mathcal{O}(\alpha)$ QED correction to the muon decay spectrum with keeping the exact dependence on the electron and the muon masses. The result is obtained by means of the standard technique. The contributions from virtual, soft, and hard photons were evaluated separately:

$$\frac{\mathrm{d}^2\Gamma^{(1)}}{\mathrm{d}x\mathrm{d}c} = \frac{\mathrm{d}^2\Gamma^{\mathrm{Virt}}}{\mathrm{d}x\mathrm{d}c} + \frac{\mathrm{d}^2\Gamma^{\mathrm{Soft}}}{\mathrm{d}x\mathrm{d}c} + \frac{\mathrm{d}^2\Gamma^{\mathrm{Hard}}}{\mathrm{d}x\mathrm{d}c}\,. \tag{8}$$

To be short, we give here only the simple formula for the soft photon contribution:

$$\frac{\mathrm{d}^2\Gamma^{\mathrm{Soft}}}{\mathrm{d}x\mathrm{d}c} = \frac{\mathrm{d}^2\Gamma^{\mathrm{Born}}}{\mathrm{d}x\mathrm{d}c}\,\delta^{\mathrm{Soft}}, \tag{9}$$

$$\delta^{\mathrm{Soft}} = -\frac{\alpha}{2\pi}\Big\{2\Big(2\ln\frac{2\Delta\varepsilon}{m_\mu} + L + \ln\frac{m_e^2}{\lambda^2}\Big)\Big[1 - \frac{1}{2\beta}l_\beta\Big] + \frac{1}{2\beta}l_\beta^2 - \frac{1}{\beta}l_\beta$$
$$+ \frac{2}{\beta}\mathrm{Li}_2\left(\frac{2\beta}{1+\beta}\right) - 2\Big\}, \qquad l_\beta = \ln\frac{1+\beta}{1-\beta}\,, \tag{10}$$

where $\Delta\epsilon$ is the maximal energy of a soft photon ($\Delta \ll 1$); λ is a fictitious photon mass; functions $\mathrm{Li}_2\,(x)$ and $\zeta(n)$ are defined in the Appendix.

The auxiliary parameters λ and $\Delta\epsilon$ cancel out in the total sum of the three contributions:

$$\frac{\mathrm{d}^2\Gamma^{(1)}}{\mathrm{d}x\mathrm{d}c} = \Gamma_0\; x^2\beta\frac{\alpha}{2\pi}\left(f_1(x) + c\xi g_1(x)\right), \tag{11}$$

$$f_1(x) = f^{\mathrm{Born}}(x)\Big(\frac{2}{\beta}A + \frac{x^2(1-\beta^2) - 4(1+x\beta)}{2x\beta}\ln\frac{q^2}{m_\mu^2}$$
$$+ \frac{4 - x^2(1-\beta^2)}{x\beta}\ln\frac{2 - x(1-\beta)}{2}\Big)$$
$$+ \frac{1}{\beta}\Big(L + 2\ln x + 2\ln\frac{1+\beta}{2}\Big)\Big\{\frac{5x^4}{384}(1-\beta^2)^3 - \frac{x^3}{4}(1-\beta^2)^2$$
$$+ \frac{3x^2}{32}(3 - 12\beta + \beta^2)(1-\beta^2) + x\Big[\frac{2}{3} + 2\beta + (1-\beta^2)\Big(\frac{3}{2} + \beta\Big)\Big]$$
$$+ \frac{1}{8}[-20 - 12\beta - 19(1-\beta^2)] + 2x + \frac{5}{6}\Big\}$$
$$+ \Big(\ln x + \ln\frac{1+\beta}{2}\Big)\Big[\frac{9}{4}x^2(1-\beta^2) + 2x(\beta^2-3) + 3\Big]$$
$$+ f^{\mathrm{Born}}(x)\Big[-\frac{11}{18}x(1-\beta^2) + \frac{22}{27}\beta^2 - \frac{2}{9}\Big]$$
$$+ x\Big(-\frac{22}{27}\beta^4 + \frac{\beta^2}{2} - \frac{11}{6}\Big) + \frac{22}{9}(3-\beta^2) - \frac{22}{3x}, \tag{12}$$

$$A = L\Big(\ln\frac{q^2}{m_\mu^2} - \ln x + \ln\frac{1+\beta}{2\beta} + \ln\frac{2 - x(1-\beta)}{2\beta}\Big) + \Big[\ln\frac{q^2}{m_\mu^2} - 2\ln x$$
$$+ 2\ln\frac{1+\beta}{2} + 4\ln\frac{2 - x(1-\beta)}{2\beta}\Big]\Big(\ln x + \ln\frac{1+\beta}{2}\Big)$$

$$+\,2\mathrm{Li}_2\left(\frac{(1-\beta)(2-x(1+\beta))}{(1+\beta)(2-x(1-\beta))}\right)-2\mathrm{Li}_2\left(\frac{2-x(1+\beta)}{2-x(1-\beta)}\right), \tag{13}$$

$$\begin{aligned}
g_1(x) = {} & g^{\mathrm{Born}}(x)\Big(\frac{2}{\beta}A-4\ln\frac{2-x(1-\beta)}{2}\Big) \\
& + \frac{1}{\beta^2}\Big(L+2\ln x+2\ln\frac{1+\beta}{2}\Big)\Big\{\frac{5x^4}{384}(1-\beta^2)^3 \\
& + \frac{x^3}{8}(1-\beta^2)^2(1-3\beta^2)+\frac{3x^2}{32}(1-\beta^2)(-11+15\beta^2-12\beta^3) \\
& + x\Big[\frac{2}{3}+2\beta+(1-\beta^2)\Big(\frac{\beta^2}{2}-2\beta+\frac{3}{2}\Big)\Big]-\frac{7}{2}-\frac{\beta}{2} \\
& + (1-\beta^2)\Big(\frac{17}{8}+\frac{\beta}{2}\Big)-\frac{1}{6x^2}\Big\}+\beta\Big(\ln x+\ln\frac{1+\beta}{2}\Big) \\
& \times\Big(\frac{9}{4}x^2(1-\beta^2)-4x+1\Big)+\frac{1}{\beta^2}\Big(\ln\frac{q^2}{m_\mu^2}-2\ln\frac{2-x(1-\beta)}{2}\Big) \\
& \times\Big\{-\frac{x^3}{48}(1-\beta^2)^2(1-19\beta^2)+x^2(1-\beta^2)\Big(-\frac{3}{2}\beta^3-\frac{5}{4}\beta^2+\frac{1}{4}\Big) \\
& + x\Big[4\beta+(1-\beta^2)\Big(-\frac{3}{4}\beta^2-4\beta-\frac{5}{4}\Big)\Big]+\frac{16}{3}-2\beta \\
& + (1-\beta^2)(2\beta-2)+\frac{1}{x}(-6+(1-\beta^2))+\frac{4}{x^2}-\frac{4}{3x^3}\Big\} \\
& + g^{\mathrm{Born}}(x)\Big[-\frac{5}{144\beta^2}x^2(1-\beta^2)^2-\frac{10}{27\beta^2}x(1-\beta^2)-\frac{55}{54}+\frac{203}{162\beta^2}\Big] \\
& + \frac{x}{81}\Big(\frac{17}{\beta}-195\beta\Big)-\frac{1}{324}\Big(\frac{595}{\beta}-1923\beta\Big)+\frac{10}{3x\beta}-\frac{1}{x^2\beta}, \\
L = {} & \ln\frac{m_\mu^2}{m_e^2}, \qquad q^2=m_\mu^2(1-x)+m_e^2.
\end{aligned} \tag{14}$$

Again the integration over the energy fraction and the angle gives us corrections to the decay width and the forward–backward asymmetry:

$$\Gamma^{(1)}=\Gamma_0\,\frac{\alpha}{2\pi}F_1(\rho), \qquad \Gamma^{(1)}_{\mathrm{FB}}=\xi\Gamma_0\,\frac{\alpha}{2\pi}G_1(\rho), \tag{15}$$

$$\begin{aligned}
F_1(\rho) = {} & (1-\rho^2)\Big(\frac{25}{4}-\frac{239}{3}\rho+\frac{25}{4}\rho^2\Big)-\rho\ln\rho\Big(20+90\rho-\frac{4}{3}\rho^2 \\
& + \frac{17}{3}\rho^3\Big)-\rho^2\ln^2\rho(36+\rho^2)-(1-\rho^2)\Big(\frac{17}{3}-\frac{64}{3}\rho+\frac{17}{3}\rho^2\Big) \\
& \times\ln(1-\rho)+4(1+30\rho^2+\rho^4)\ln\rho\ln(1-\rho)+6(1+16\rho^2+\rho^4)
\end{aligned}$$

$$\times [\mathrm{Li}_2(\rho) - \zeta(2)] + 64\rho^{3/2}(1+\rho)\Big(3\zeta(2) - 2\mathrm{Li}_2(\sqrt{\rho})
+ 2\mathrm{Li}_2(-\sqrt{\rho}) - \ln\rho \ln\frac{1-\sqrt{\rho}}{1+\sqrt{\rho}}\Big), \tag{16}$$

$$\begin{aligned}
G_1(\rho) = &-\frac{472}{27}(1-\sqrt{\rho}) + (1-\rho)\Big(\frac{1271}{108} + \frac{47}{27}\rho^{1/2} - \frac{2959}{108}\rho + 60\rho^{3/2} \\
&- \frac{4657}{108}\rho^2 + 11\rho^{5/2} - \frac{21}{4}\rho^3\Big) - \ln(1+\sqrt{\rho})(1-\rho)\Big(\frac{100}{9} + \frac{52}{9}\rho \\
&+ \frac{268}{9}\rho^2 + \frac{20}{3}\rho^3\Big) + \ln(1-\sqrt{\rho})(1-\sqrt{\rho})\frac{2048}{9} \\
&+ \ln(1-\sqrt{\rho})(1-\rho)\Big(-\frac{2035}{9} + \frac{2048}{9}\sqrt{\rho} - \frac{1987}{9}\rho + \frac{512}{3}\rho^{3/2} \\
&- \frac{637}{9}\rho^2 + \frac{17}{3}\rho^3\Big) - \ln\rho(1-\sqrt{\rho})\frac{608}{9} + \ln\rho(1-\rho)\Big(\frac{608}{9} \\
&- \frac{614}{9}\sqrt{\rho} + \frac{623}{9}\rho - 56\rho^{3/2} + \frac{967}{18}\rho^2 + 2\rho^{5/2} + \frac{1}{2}\rho^3\Big) \\
&+ \ln\rho[\ln(1-\sqrt{\rho}) + 2\ln(1+\sqrt{\rho})]\Big(\frac{2}{3} + 8\rho^2 - \frac{32}{3}\rho^3 + 2\rho^4\Big) \\
&+ \rho^2\ln^2\rho\Big(7 + 16\rho - \frac{3}{2}\rho^2\Big) + (\zeta(2) + 2\mathrm{Li}_2(-\sqrt{\rho}))\Big(\frac{14}{3} + 16\rho \\
&+ 32\rho^2 - \frac{16}{3}\rho^3 + 6\rho^4\Big) - \rho^2\frac{2}{3}\ln^3\rho + 16\rho^2\ln\rho[2\zeta(2) - 2\mathrm{Li}_2(-\sqrt{\rho}) \\
&- \mathrm{Li}_2(\sqrt{\rho})] + 16\rho^2[2\mathrm{Li}_3(\sqrt{\rho}) + 12\mathrm{Li}_3(-\sqrt{\rho}) + 7\zeta(3)].
\end{aligned} \tag{17}$$

The definitions of the special functions, used above, are given in the Appendix.

The expression for $F_1(\rho)$ coincides with the one, received in Ref. [14], starting from a differential distribution of a different kind. We reproduced here the formula for the sake of completeness and for a comparison with $G_1(\rho)$. Both the functions $F_1(\rho)$ and $G_1(\rho)$ are vanishing in the limit $\rho \to 1$, because of the vanishing phase space volume.

It is interesting to note, that $G_1(\rho)$ contains terms of the first power in the electron and the muon mass ratio, while $F_1(\rho)$ does not (see the discussion about the odd mass terms in Ref. [9]). The linear mass terms in $G_1(\rho)$ can be clearly seen from the expansion[3] of the exact formula:

$$\begin{aligned}
G_1 = &-\frac{617}{108} + \frac{14}{3}\zeta(2) - \frac{8}{3}\sqrt{\rho} + \rho\Big(-32 + 16\zeta(2) + \frac{2}{3}\ln\rho\Big) \\
&+ \rho^{3/2}\Big(\frac{568}{27} + \frac{112}{9}\ln\rho\Big) + \rho^2\Big(\frac{281}{6} + 32\zeta(2) + 112\zeta(3) - 16\ln\rho
\end{aligned}$$

[3] The expansion of $F_1(\rho)$ can be found in Ref.[14].

$$
\begin{aligned}
&+32\zeta(2)\ln\rho+7\ln^2\rho-\frac{2}{3}\ln^3\rho\Big)+\rho^{5/2}\Big(-\frac{95624}{225}+\frac{1232}{15}\ln\rho\Big)\\
&+\rho^3\Big(\frac{5662}{27}-\frac{16}{3}\zeta(2)-\frac{698}{9}\ln\rho+16\ln^2\rho\Big)\\
&+\rho^{7/2}\Big(-\frac{134248}{4725}-\frac{512}{63}\ln\rho\Big)+\mathcal{O}(\rho^4).
\end{aligned}
\tag{18}
$$

One can check that the term with $|m_e/m_\mu|^1 = \sqrt{\rho}$ is coming from the integration over the region of large (close to the upper limit) values of the electron energy fraction. The absolute value appears from the integration, it can be important for analytical continuations. Analogous terms with the first order mass ratio have been found in the forward–backward asymmetry in the process of electron–positron annihilation into heavy leptons [15].

The incorporation the first order correction gives the following formula for the spectrum:

$$
\frac{\mathrm{d}^2\Gamma^{\mathrm{rad.corr.}}}{\mathrm{d}x\mathrm{d}c}=\frac{\mathrm{d}^2\Gamma^{\mathrm{Born}}}{\mathrm{d}x\mathrm{d}c}+\frac{\mathrm{d}^2\Gamma^{(1)}}{\mathrm{d}x\mathrm{d}c}+\mathcal{O}(\alpha^2).
\tag{19}
$$

4 Comparisons and numerical results

The agreement with the massless formulae [16] for $f_1(x)$ and $g_1(x)$ is checked. Our calculations for the massive case agree in part with the known results. Namely, function $f_1(x)$ coincides with the result of Ref. [17], where a mistake should be corrected according to Ref. [16] (see also Appendix C in Ref. [9]). The same function does agree with the one given in Ref. [18], while the the integral over the real photon phase volume has not been taken analytically there. Moreover, a confirmation of the given formula for $f_1(x)$ comes from the comparison of the result of its integration over the energy fraction with Ref. [14], as mentioned above.

Function $g_1(x)$, which describes the polarized part of the decay spectrum, could have been calculated long time ago on the same basis as $f_1(x)$. Nevertheless, our results for $g_1(x)$ and $G_1(\rho)$ are new. A partial comparison between the virtual (loop) diagram contributions shows an agreement with the corresponding quantity from the calculation of the top–quark decay spectrum [19] [4].

In Table 1 we show the effect of the mass terms at the lowest (Born) level and at the level of the first order correction for different points of the energy spectrum;

[4] Because of different choices of observables, the comparison of the total spectra is impossible.

Table 1
The effect of the finite electron–muon mass ratio versus the electron energy fraction.

x	$h_0(x)$	$h^{\rm Born}(\tilde{x})$	$\delta_m^{\rm Born}$	$h_1^{m_e\to 0}(x)$	$h_1(\tilde{x})$	$\delta_m^{(1)}$
$c=0,\quad \xi=1$						
0.05	0.00725	0.00711	−194.5	4.11481	4.10454	−16.45
0.1	0.02800	0.02786	−49.52	5.95508	5.95444	−0.266
0.2	0.10400	0.10387	−12.80	8.68399	8.68517	0.132
0.3	0.21600	0.21587	−5.796	10.3054	10.3067	0.071
0.5	0.50000	0.49989	−2.105	8.66761	8.66871	0.026
0.7	0.78400	0.78391	−1.088	−1.55489	−1.55410	0.012
0.9	0.97200	0.97193	−0.742	−25.6678	−25.6670	0.010
0.99	0.99970	0.99963	−0.702	−67.6027	−67.6011	0.019
0.999	1.00000	0.99993	−0.702	−107.665	−107.663	0.026
$c=1,\quad \xi=1$						
0.05	0.00950	0.00928	−236.8	3.6880	3.6657	−27.32
0.1	0.03600	0.03579	−59.02	5.2896	5.2850	−1.490
0.2	0.12800	0.12781	−14.51	7.4177	7.4179	0.027
0.3	0.25200	0.25184	−6.186	7.8913	7.8925	0.054
0.5	0.50000	0.49991	−1.871	2.7579	2.7589	0.024
0.7	0.58800	0.58796	−0.659	−8.1130	−8.1129	0.001
0.9	0.32400	0.32400	−0.150	−13.120	−13.121	−0.022
0.99	0.03920	0.03920	−0.013	−3.2805	−3.2807	−0.053
0.999	0.00399	0.00399	−0.001	−0.4949	−0.4949	−0.081

$$\delta_m^{\rm Born} = 10^4\cdot\left(\frac{h^{\rm Born}(\tilde{x})}{h_0(x)} - 1\right), \qquad \delta_m^{(1)} = 10^4\cdot\frac{\alpha}{2\pi}\,\frac{h_1(\tilde{x}) - h_1^{m_e\to 0}(x)}{h_0(x)},$$
$$h_{0,1}(x) = f_{0,1}(x) + c\xi g_{0,1}(x), \qquad h^{\rm Born}(\tilde{x}) = f^{\rm Born}(\tilde{x}) + c\xi g^{\rm Born}(\tilde{x}), \quad (20)$$

where the functions $f_1^{m_e\to 0}(x)$ and $g_1^{m_e\to 0}(x)$ can be received by applying $m_e = 0$ in the general expressions for $f_1(x)$ and $g_1(x)$ everywhere, except the argument of the large logarithm L. In practice, the massless functions can be taken from Ref. [16]. The argument of the functions with the exact mass dependence is rescaled: $\tilde{x} = x\cdot x_{\rm max}$. One can see, that the effect of the mass terms in the $\mathcal{O}(\alpha)$ order is below the 10^{-4} precision tag (see $\delta_m^{(1)}$) for experimentally [1,2] preferable values of the electron energy fraction $x > 0.3$.

In Table 2 we show the effect of the mass terms in the integrated quantities:

Table 2
The effect of the finite mass ratio in the total decay width and in the forward–backward asymmetry.

ρ	$F(0)$	$F(\rho)$	$\delta_F^{\rm Born}$	$F_1(0)$	$F_1(\rho)$	$\delta_F^{(1)}$
$\frac{m_e^2}{m_\mu^2} = 2.34 \cdot 10^{-5}$	1.0000	0.9998	−1.871	−3.6196	−3.6152	0.051
$\frac{m_\mu^2}{m_\tau^2} = 3.54 \cdot 10^{-3}$	1.0000	0.9726	−274.3	−3.6196	−3.3367	3.286
ρ	$G(0)$	$G(\rho)$	$\delta_G^{\rm Born}$	$G_1(0)$	$G_1(\rho)$	$\delta_G^{(1)}$
$\frac{m_e^2}{m_\mu^2} = 2.34 \cdot 10^{-5}$	−0.3333	−0.3333	−0.036	1.9634	1.9502	0.460
$\frac{m_\mu^2}{m_\tau^2} = 3.54 \cdot 10^{-3}$	−0.3333	−0.3314	−56.71	1.9634	1.7651	6.908

$$
\begin{aligned}
\delta_F^{\rm Born} &= 10^4 \cdot \left(\frac{F(\rho)}{F(0)} - 1 \right), \qquad & \delta_F^{(1)} &= 10^4 \cdot \frac{\alpha}{2\pi} \, \frac{F_1(\rho) - F_1(0)}{F(0)}, \\
\delta_G^{\rm Born} &= 10^4 \cdot \left(\frac{G(\rho)}{G(0)} - 1 \right), \qquad & \delta_G^{(1)} &= 10^4 \cdot \frac{\alpha}{2\pi} \, \frac{G_1(\rho) - G_1(0)}{G(0)}. \qquad (21)
\end{aligned}
$$

The effect due to finite electron mass in function $G_1(\rho)$ is approaching the 10^{-4} level (for muon decay).

5 Conclusions

The present calculations can be easily extended for the case, where the final electron polarization is measured. But, for the moment, the experimental precision there does not call for an account of small mass terms in the theoretical predictions.

In this way, we considered the first order QED radiative correction to the muon decay spectrum. Formulae for the polarized part of the spectrum ($g_1(x)$ and G_1) with the exact dependence on the electron mass are received for the first time. The results presented are relevant for modern and future precise measurements of the muon decay spectrum. Our formulae are valid for the leptonic τ decays as well.

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Appendix
Definition of special functions

The Riemann ζ-function and the polilogarithm functions are defined as usually:

$$\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) = 1.20205690315959\ldots$$
$$\mathrm{Li}_2\,(y) = -\int\limits_0^y \mathrm{d}x\, \frac{\ln(1-x)}{x}\,, \qquad \mathrm{Li}_3\,(y) = \int\limits_0^y \mathrm{d}x\, \frac{\mathrm{Li}_2\,(x)}{x}\,. \tag{A.1}$$

The following identities can help in numerical evaluations and estimates:

$$\mathrm{Li}_2\,(1) = \zeta(2), \qquad \mathrm{Li}_2\,(-1) = -\frac{1}{2}\zeta(2), \qquad \mathrm{Li}_3\,(1) = \zeta(3),$$
$$\mathrm{Li}_3\,(-1) = -\frac{3}{4}\zeta(3), \qquad \mathrm{Li}_2\left(y^2\right) = 2(\mathrm{Li}_2\,(y) + \mathrm{Li}_2\,(-y)),$$
$$\mathrm{Li}_3\left(y^2\right) = 4(\mathrm{Li}_3\,(y) + \mathrm{Li}_3\,(-y)). \tag{A.2}$$

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