Virtual and Soft Pair Corrections to Polarized Muon Decay Spectrum

A.B. Arbuzov [1]

Bogoliubov Laboratory of Theoretical Physics,
JINR, Dubna, 141980 Russia
e-mail: *arbuzov@thsun1.jinr.ru*

Abstract

Radiative corrections to the muon decay spectrum due to soft and virtual electron–positron pairs are calculated.

Key words: muon decay, radiative corrections

1 Introduction

The experiment $\mathcal{TWIST}$ [1,2] is currently running at Canada's National Laboratory TRIUMF. It is going to measure the muon decay spectrum [3,4] with the accuracy level of about $1 \cdot 10^{-4}$. That will make a serious test of the space–time structure of the weak interaction. The experiment is able to put stringent limits on a bunch of parameters in models beyond the Standard Model (SM), *e.g.*, on the mass and the mixing angle of a possible right–handed W-boson. To confront the experimental results with the SM, adequately accurate theoretical predictions should be provided. This requires to calculate radiative corrections within the perturbative Quantum Electrodynamics (QED). Here we will present analytical results for two specific contributions, related to radiation of virtual and soft real electron–positron pairs. The corrections under consideration are of the order $\mathcal{O}\left(\alpha^2\right)$, where α is the fine structure constant.

The contributions of virtual $\mu^+\mu^-$, $\tau^+\tau^-$ and hadronic pairs were found [5] to be small compared with the $1 \cdot 10^{-4}$ precision tag of the modern experiments. The contribution of e^+e^- pairs is enhanced by powers of the large logarithm

[1] A certain part of this work was performed in University of Alberta, Edmonton, Canada

$L = \ln(m_\mu^2/m_e^2) \approx 10.66$. Analysis of the leading and next–to–leading terms from this correction in Refs. [6,7] has shown that the numerical effect is not as small as for other leptonic flavors, and it should be taken into account. Comparison of the leading and next–to–leading contributions revealed a poor convergence of the series in L. Calculation of the terms without the large logarithm was found to be desirable.

Within the Standard Model, the differential distribution of electrons (summed over electron spin states) in the polarized muon decay can be represented as

$$\frac{\mathrm{d}^2\Gamma^{\mu^\mp \to e^\mp \nu\bar{\nu}}}{\mathrm{d}x\mathrm{d}c} = \Gamma_0 \left[F(x) \pm cP_\mu G(x)\right], \qquad \Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3},$$
$$c = \cos\theta, \qquad x = \frac{2m_\mu E_e}{m_\mu^2 + m_e^2}, \qquad x_0 \le x \le 1, \qquad x_0 = \frac{2m_\mu m_e}{m_\mu^2 + m_e^2}, \qquad (1)$$

where m_μ and m_e are the muon and electron masses; G_F is the Fermi coupling constant; θ is the angle between the muon polarization vector $\vec{P}_\mu$ and the electron (or positron) momentum; E_e and x are the energy and the energy fraction of $e^\pm$. Here we adopt the definition of the Fermi coupling constant following Ref. [8]. Functions $F(x)$ and $G(x)$ describe the isotropic and anisotropic parts of the spectrum, respectively. Within perturbative QED, they can be expanded in series in α:

$$F(x) = f_{\mathrm{Born}}(x) + \frac{\alpha}{2\pi} f_1(x) + \left(\frac{\alpha}{2\pi}\right)^2 f_2(x) + \left(\frac{\alpha}{2\pi}\right)^3 f_3(x) + \mathcal{O}\left(\alpha^4\right), \quad (2)$$

and in the same way for $G(x)$. Among different contributions into the functions $F(x)$ and $G(x)$ (see Ref. [6] for details and discussion), there are ones related to electron–positron pair production. In this Letter we will consider the effect of soft and virtual e^+e^- pairs.

2 Soft e^+e^- Pairs

The process of real pair production does not reveal any infrared singularity, contrary to the case of photon radiation. Nevertheless, a separate consideration of soft pair emission can be of interest. In fact, e^+e^- pairs with energy below a certain threshold can't be observed in experiments with muons decaying at rest. So, the corresponding contribution is a specific correction to the measured decay spectrum. Moreover, the behavior of the real pair emission in the soft limit is not smooth. An integration over the domain between the threshold of real pair production and a certain cut on the maximal energy of the soft pair is desirable.

The maximal energy of the soft pair is assumed to be large compared with the electron mass:

$$E^{\rm pair} \le \Delta\frac{m_\mu}{2}, \qquad \frac{m_e}{m_\mu} \ll \Delta \ll 1. \tag{3}$$

Due to the smallness of the pair component energies, the matrix element M of the process

$$\mu^-(p) \;\longrightarrow e^-(q) \;+\; \nu_\mu(r_1) \;+\; \bar{\nu}_e(r_2) \;+\; e^+(p_+) \;+\; e^-(p_-) \tag{4}$$

can be expressed as a product of the matrix element M_0 of the hard sub–process (the non–radiative muon decay) and the classic accompanying radiation factor:

$$M = M_0\frac{4\pi\alpha}{k^2}\bar{v}(p_+)\gamma^\mu u(p_-)J_\mu, \qquad k = p_+ + p_-, \tag{5}$$

where $p_{+,-}$ are the momenta of the positron and electron from the created pair. The radiation factor reads

$$J_\mu = \frac{p_\mu}{pk - \frac{1}{2}k^2} - \frac{q_\mu}{qk + \frac{1}{2}k^2}\,. \tag{6}$$

Performing the covariant integration of the summed over spin states modulus of the matrix element over the pair components momenta, we obtain

$$\begin{aligned}
&\sum_{\rm spin}|\bar{v}(p_+)\gamma^\mu u(p_-)|^2 = 4(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - \frac{k^2}{2}\,g^{\mu\nu}),\\
&\int\frac{{\rm d}^3\boldsymbol{p}_+{\rm d}^3\boldsymbol{p}_-}{p_+^0p_-^0}\delta^4(p_+ + p_- - k)(p_+^\mu p_-^\nu + p_+^\nu p_-^\mu - \frac{k^2}{2}g^{\mu\nu}) =\\
&\qquad = \Big(-\frac{2\pi}{3}(k^2+2m_e^2)\sqrt{1-\frac{4m_e^2}{k^2}}\,\Big)(g^{\mu\nu} - \frac{1}{k^2}k^\mu k^\nu).
\end{aligned} \tag{7}$$

It is convenient to parameterize the phase volume of the total pair momentum as

$${\rm d}^4k = {\rm d}k_0\boldsymbol{k}^2{\rm d}|\boldsymbol{k}|{\rm d}\Omega_k = \pi{\rm d}k_0{\rm d}k^2\,\sqrt{k_0^2-k^2}\;{\rm d}c_k\,, \tag{8}$$

where a trivial integration over the azimuthal angle was performed. Now I integrate over the total pair momentum with the condition (3) ($k_0 \equiv E^{\rm pair}$). In this way I got the following result for the soft pair contribution:

$$
\begin{aligned}
&\frac{\mathrm{d}\Gamma^{\mathrm{SP}}}{\mathrm{d}c\ \mathrm{d}x} = \frac{\mathrm{d}\Gamma^{\mathrm{Born}}}{\mathrm{d}c\ \mathrm{d}x}\delta^{\mathrm{SP}}, \qquad \frac{\mathrm{d}\Gamma^{\mathrm{Born}}}{\mathrm{d}c\ \mathrm{d}x} = \Gamma_0\left[f_0(x) \pm cP_\mu g_0(x)\right] + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right), \\
&f_0(x) = x^2(3-2x), \qquad g_0(x) = x^2(1-2x), \\
&\delta^{\mathrm{SP}} = \frac{\alpha^2}{3\pi^2}\biggl[\frac{1}{12}\ln^3 A - \frac{2}{3}\ln^2 A + \ln A\biggl(\frac{61}{18} - \zeta(2)\biggr) - \frac{223}{27} \\
&\qquad + \frac{8}{3}\zeta(2) + 2\zeta(3)\biggr], \qquad\qquad (9) \\
&\ln A = L + 2\ln\Delta, \qquad \zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}, \qquad \zeta(2) = \frac{\pi^2}{6}\,.
\end{aligned}
$$

So we calculated explicitly all the terms in δ^{SP} except the ones suppressed by the small factors $(\alpha/\pi)^2 m_e^2/m_\mu^2$ and $(\alpha/\pi)^2\Delta$.

3 Virtual e^+e^- Pair

We will use here the substitution suggested by J. Schwinger for the photon propagator (with 4–momentum k) corrected by a one–loop vacuum polarization insertion:

$$
\begin{aligned}
&\frac{1}{k^2 - \lambda^2 + i0} \to \frac{\alpha}{\pi}\int\limits_0^1 \frac{\mathrm{d}v\phi(v)}{1-v^2}\,\frac{1}{k^2 - M^2 + i0}\,, \quad M^2 = \frac{4m_2^2}{1-v^2}\,, \qquad (10) \\
&\quad \phi(v) = \frac{2}{3} - \frac{1}{3}(1-v^2)(2-v^2),
\end{aligned}
$$

where m_2 is the mass of the fermion in the loop.

Using the standard technique of integration over Feynman parameters and the on–mass–shell scheme for renormalization of the ultra–violet singularity [9], I got the following result for the virtual e^+e^- pair contribution:

$$
\frac{\Gamma^{\mathrm{VP}}}{\mathrm{d}c\ \mathrm{d}x} = \Gamma_0\biggl(\frac{\alpha}{2\pi}\biggr)^2\left[f_{2,\mathrm{virt}}^{(e^+e^-)}(x) \pm cP_\mu g_{2,\mathrm{virt}}^{(e^+e^-)}(x) + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right)\right], \qquad (11)
$$

where

$$
\begin{aligned}
f_{2,\mathrm{virt}}^{(e^+e^-)}(x) &= f_0(x)W(x) - 2x^2\ln xL - 2x^2\ln^2 x - 2x^2\mathrm{Li}_2\,(1-x) \\
&\quad - \frac{2}{3(1-x)}\ln x + \frac{2}{3}x\ln x + 7x^2\ln x + \frac{2}{3}\ln x, \\
g_{2,\mathrm{virt}}^{(e^+e^-)}(x) &= g_0(x)W(x) - \frac{2}{3}x^2\ln xL - \frac{2}{3}x^2\ln^2 x - \frac{2}{3}x^2\mathrm{Li}_2\,(1-x) \qquad (12)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{2}{3(1-x)}\ln x - \frac{2}{3}x\ln x + \frac{13}{9}x^2\ln x - \frac{2}{3}\ln x,\\
W(x) &= -\frac{1}{9}L^3 + \Big(\frac{25}{18} - \frac{2}{3}\ln x\Big)L^2 + \Big(-\frac{397}{54} - \frac{4}{3}\zeta(2) + \frac{38}{9}\ln x\\
&-\frac{4}{3}\ln^2 x - \frac{4}{3}\mathrm{Li}_2\left(1-x\right)\Big)L + \frac{517}{27} - \frac{8}{3}\zeta(2)\ln x + \frac{22}{9}\zeta(2)\\
&+\frac{4}{3}\zeta(3) - \frac{8}{3}\ln x\mathrm{Li}_2\left(1-x\right) - \frac{265}{27}\ln x + \frac{38}{9}\ln^2 x - \frac{8}{9}\ln^3 x\\
&+\frac{38}{9}\mathrm{Li}_2\left(1-x\right) - \frac{8}{3}\mathrm{S}_{1,2}\left(1-x\right) + \frac{4}{3}\mathrm{Li}_3\left(1-x\right), \qquad (13)\\
\mathrm{Li}_2\left(x\right) &\equiv -\int\limits_0^x \mathrm{d}y\, \frac{\ln(1-y)}{y}, \qquad \mathrm{Li}_3\left(x\right) \equiv \int\limits_0^x \mathrm{d}y\, \frac{\mathrm{Li}_2\left(y\right)}{y},\\
\mathrm{S}_{1,2}\left(x\right) &\equiv \frac{1}{2}\int\limits_0^x \mathrm{d}y\, \frac{\ln^2(1-y)}{y}.
\end{aligned}
$$

It is worth to note that the sub–leading virtual corrections don't factorize before the Born functions $f_0(x)$ and $g_0(x)$. Such a situation happens in the first order virtual photonic corrections too.

By integration over the energy fraction and the angle we receive the corresponding contribution to the total muon width:

$$
\begin{aligned}
\Gamma^{\mathrm{VP}} &= \int\limits_{-1}^{1}\mathrm{d}c\int\limits_0^1 \mathrm{d}x\, \frac{\Gamma^{\mathrm{VP}}}{\mathrm{d}c\,\mathrm{d}x} = \Gamma_0\Big(\frac{\alpha}{2\pi}\Big)^2\Big[-\frac{1}{9}L^3 + \frac{5}{3}L^2 - \Big(\frac{265}{36} + \frac{8}{3}\zeta(2)\Big)L\\
&+ \frac{20063}{1296} + \frac{61}{9}\zeta(2) + \frac{16}{3}\zeta(3)\Big] \approx -5.0497\cdot 10^{-5}\,\Gamma_0. \qquad (14)
\end{aligned}
$$

This quantity was calculated earlier in Ref. [10] by numerical integration using dispersion relations:

$$
\Gamma^{\mathrm{VP}}([10]) \approx -5.1326\cdot 10^{-5}\,\Gamma_0, \qquad (15)
$$

which is close but different from my number (14). The reason for this difference will be investigated elsewhere. At least part of it can be due to terms proportional to $(\alpha/\pi)^2(m_e^2/m_\mu^2)L^n$, which were omitted in my calculation.

The correction to the forward–backward asymmetry of the decay can be found also:

$$
\Gamma^{\mathrm{VP}}_{\mathrm{FB}} = \Bigg[\int\limits_0^1\mathrm{d}c - \int\limits_{-1}^{0}\mathrm{d}c\Bigg]\int\limits_0^1\mathrm{d}x\,\frac{\Gamma^{\mathrm{VP}}}{\mathrm{d}c\,\mathrm{d}x} = \Gamma_0\Big(\frac{\alpha}{2\pi}\Big)^2\Big[\frac{1}{54}L^3 - \frac{13}{54}L^2 + \Big(\frac{647}{648}
$$

$$+\frac{4}{9}\zeta(2)\Big)L - \frac{10339}{7776} - \frac{3}{2}\zeta(2) - \frac{8}{9}\zeta(3)\Big] \approx -1.17\cdot 10^{-5}\,\Gamma_0. \tag{16}$$

4 Numerical Results and Conclusions

The relative effect of the soft pair correction depends only on the cut value. It is shown in Fig. 1. The soft pair approximation (3) is not valid for values

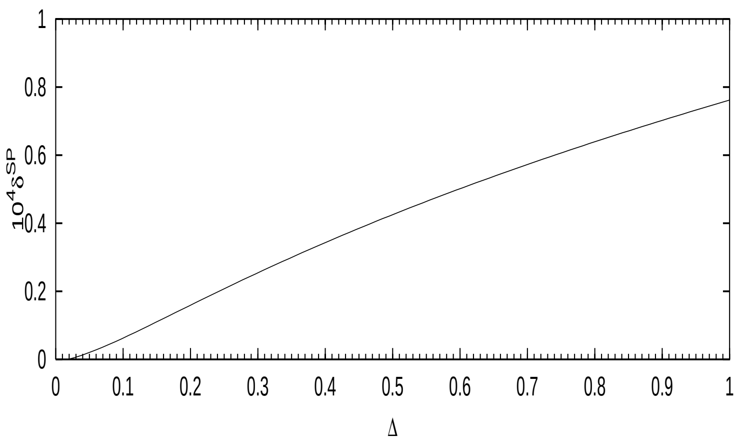

Fig. 1. The relative effect of soft pair corrections *versus* the cut value.

of Δ close to the threshold of real pair production and for large $\Delta \sim 1$. But it can be used there as a simple estimate. So, by taking $\Delta = 1$ we make an estimate of the order of magnitude of the total contribution due to real e^+e^- pairs (here the estimate is about two times the true value). For very small values of Δ the correction should vanish in any case, so the approximation is really safe there.

Let us define the relative contribution of the virtual e^+e^- pair corrections in the form

$$\delta^{\mathrm{VP}}(x) = \left(\frac{\alpha}{2\pi}\right)^2 \frac{f_{2,\mathrm{virt}}^{(e^+e^-)}(x) + cP_\mu g_{2,\mathrm{virt}}^{(e^+e^-)}(x)}{f_0(x) + cP_\mu g_0(x)}\,. \tag{17}$$

The dependence of this function on the electron energy fraction is shown in Fig. 2 in different approximations for $P_\mu = 1$, $c = 1$. The dependence on c is very weak, because the main part of the correction is factorized before the Born–level functions. The leading logarithmic (LL) approximation takes into account only the terms of the order $\mathcal{O}\left(\alpha^2 L^{3,2}\right)$, the next–to–leading logarithmic (NLL) approximation includes also the $\mathcal{O}\left(\alpha^2 L^1\right)$ terms, and the next–to–next–to–leading approximation (NNL) represents the complete result.

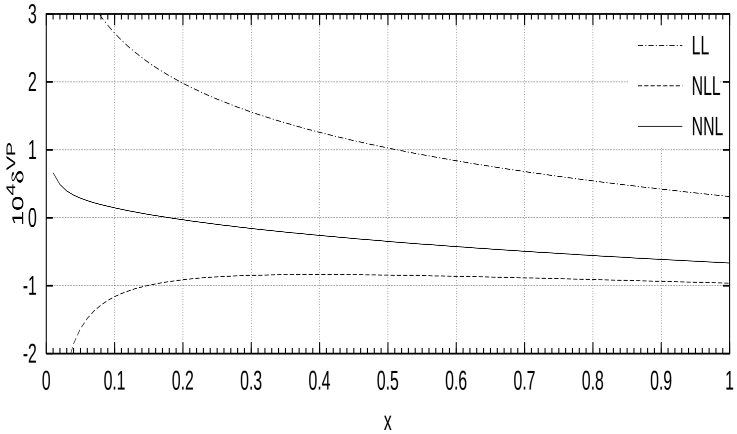

Fig. 2. The relative effect of virtual pair corrections *versus* electron energy fraction in different approximations.

The third power of the large logarithm cancels out in the sum of the virtual and soft pair contributions:

$$\frac{\Gamma^{\mathrm{SVP}}}{\mathrm{d}c\,\mathrm{d}x} = \Gamma_0\left(\frac{\alpha}{2\pi}\right)^2\left[f_{2,\mathrm{SV}}^{(e^+e^-)}(x) \pm cP_\mu g_{2,\mathrm{SV}}^{(e^+e^-)}(x) + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2},\,\Delta\right)\right], \tag{18}$$

where

$$\begin{aligned}
f_{2,\mathrm{SV}}^{(e^+e^-)}(x) &= f_0(x)U(x) - 2x^2\ln xL - 2x^2\ln^2 x - 2x^2\mathrm{Li}_2\left(1-x\right) \\
&\quad - \frac{2}{3(1-x)}\ln x + \frac{2}{3}x\ln x + 7x^2\ln x + \frac{2}{3}\ln x, \\
g_{2,\mathrm{SV}}^{(e^+e^-)}(x) &= g_0(x)U(x) - \frac{2}{3}x^2\ln xL - \frac{2}{3}x^2\ln^2 x - \frac{2}{3}x^2\mathrm{Li}_2\left(1-x\right) \\
&\quad + \frac{2}{3(1-x)}\ln x - \frac{2}{3}x\ln x + \frac{13}{9}x^2\ln x - \frac{2}{3}\ln x, \\
U(x) &= \left(\frac{1}{2} + \frac{2}{3}\ln\Delta - \frac{2}{3}\ln x\right)L^2 + \left(\frac{4}{3}\ln^2\Delta - \frac{32}{9}\ln\Delta\right. \\
&\quad \left. - \frac{4}{3}\mathrm{Li}_2\left(1-x\right) - \frac{4}{3}\ln^2 x + \frac{38}{9}\ln x - \frac{17}{6} - \frac{8}{3}\zeta(2)\right)L \\
&\quad + \frac{8}{9}\ln^3\Delta - \frac{32}{9}\ln^2\Delta - \frac{8}{3}\zeta(2)\ln\Delta + \frac{244}{27}\ln\Delta \\
&\quad + \frac{4}{3}\mathrm{Li}_3\left(1-x\right) - \frac{8}{3}\mathrm{S}_{1,2}\left(1-x\right) - \frac{8}{9}\ln^3 x - \frac{8}{3}\ln x\mathrm{Li}_2\left(1-x\right) \\
&\quad + \frac{38}{9}\mathrm{Li}_2\left(1-x\right) + \frac{38}{9}\ln^2 x - \frac{8}{3}\zeta(2)\ln x - \frac{265}{27}\ln x \\
&\quad + \frac{659}{81} + 6\zeta(2) + 4\zeta(3).
\end{aligned} \tag{19}$$

I checked that the leading and next–to–leading terms in the above formula agree with the corresponding contribution obtained within the fragmentation function formalism in Refs. [6,7].

In this way we simulate the experimental set–up with a certain energy threshold for registration of pairs, while events with pair production above the threshold (with several visible charged particles in the final state) are rejected.

If the radiation of real pairs is completely forbidden by kinematics (or experimental conditions), only the virtual corrections (12) contribute. That happens, for instance at large values of $x \gtrsim 0.99$.

Thus, two contributions to the total set of radiative corrections for the muon decay spectrum are presented. They are required to reach the level of the theoretical accuracy below $1 \cdot 10^{-4}$. The formulae can be used for semi–analytical estimates and as a part of a Monte Carlo code to describe the pair production contribution to the decay spectrum.

Acknowledgements

I am grateful to V. Bytev and E. Kuraev for discussions. This work was supported by RFBR grant 03-02-16731.

Appendix A
Asymptotic expressions for the muon form factor

Using the Schwinger substitution (10), I reproduced the known [12,13] asymptotic expressions for the $\mathcal{O}\left(\alpha^2\right)$ virtual pair contributions into the Dirac form factor of muon:

$$
\begin{aligned}
& F_1^{(4,a)}(m_1, m_2, Q^2)\Big|_{m_1, m_2 \ll Q^2} = \left(\frac{\alpha}{\pi}\right)^2 \left(\frac{e_2}{e}\right)^2 \biggl\{ -\frac{1}{36}L^3 + \frac{19}{72}L^2 \\
& \quad - \left(\frac{265}{216} + \frac{\zeta(2)}{6}\right)L + D\left(\frac{m_1}{m_2}\right)\biggr\}, \qquad \text{(A.1)} \\
& D(1) = \frac{383}{108} - \frac{1}{4}\zeta(2), \\
& D(0) = \frac{3355}{1296} + \frac{19}{36}\zeta(2) - \frac{1}{3}\zeta(3), \\
& D(R)\Big|_{R \gg 1} = \frac{1}{36}l^3 - \frac{13}{72}l^2 + \left(\frac{133}{216} + \frac{\zeta(2)}{3}\right)l + \frac{67}{54} - \frac{7}{36}\zeta(2) - \frac{1}{3}\zeta(3),
\end{aligned}
$$

$$L \equiv \ln \frac{Q^2}{m_2^2}, \qquad l \equiv \ln R = \ln \frac{m_1^2}{m_2^2},$$

where $m_1 = m_\mu$ is the muon mass; m_2 is the mass of the fermion in the loop; e and e_2 is the muon and fermion charges, respectively; $-Q^2$ is the square of the momentum transferred in the spacelike region: $-Q^2 = (p_1 - p_2)^2 < 0$, where p_1 and p_2 are the initial and the final muon four–momenta.

References

[1] N. L. Rodning *et al.*, Nucl. Phys. Proc. Suppl. **98** (2001) 247.

[2] M. Quraan *et al.*, Nucl. Phys. **A663** (2000) 903.

[3] W. Fetscher and H. J. Gerber, Eur. Phys. J. C **15** (2000) 316.

[4] Y. Kuno and Y. Okada, Rev. Mod. Phys. **73** (2001) 151 [hep-ph/9909265].

[5] A. I. Davydychev, K. Schilcher and H. Spiesberger, Eur. Phys. J. C **19**, 99 (2001).

[6] A. Arbuzov, hep-ph/0206036.

[7] A. Arbuzov and K. Melnikov, Phys. Rev. D **66** (2002) 093003.

[8] T. van Ritbergen and R. G. Stuart, Nucl. Phys. B **564** (2000) 343.

[9] S.M. Berman and A. Sirlin, Ann. Phys. **20** (1962) 20.

[10] T. van Ritbergen and R. G. Stuart, Phys. Rev. Lett. **82** (1999) 488.

[11] A. I. Davydychev and V. A. Smirnov, hep-ph/0210171.

[12] G. J. Burgers, Phys. Lett. B **164** (1985) 167.

[13] A. H. Hoang, J. H. Kuhn and T. Teubner, Nucl. Phys. B **452** (1995) 173.