

Hadronic corrections at $\mathcal{O}(\alpha^2)$ to the energy spectrum of muon decay^{*}

A.I. Davydychev^{a,b}, K. Schilcher, H. Spiesberger^c

Institut für Physik, Johannes-Gutenberg-Universität, Staudinger Weg 7, 55099 Mainz, Germany

Received: 24 November 2000 / Published online: 5 February 2001 – © Springer-Verlag 2001

Abstract. We consider the impact of $\mathcal{O}(\alpha^2)$ hadronic corrections to the energy spectrum of the decay electron in muon decay. We find that the correction can be described, within good approximation, by a linear function in the electron energy. Explicit expressions for the form factors needed in an approach based on dispersion integrals are given.

1. Testing the electroweak sector of the Standard Model requires to fix the coupling constants of the Lagrangian by relating a number of high-precision experiments to theoretical predictions. Ideally, experimental data and theoretical predictions should be known with comparable precision. Apart from the electromagnetic coupling constant α and the Z -boson mass m_Z , the input of choice is the muon life time τ_μ , with the present best value $\tau_\mu = (2.19703 \pm 0.00004) \mu s$ [1]. Experiments are planned at the Paul Scherrer Institute [2] and the Rutherford Appleton Laboratory [3] which would reduce the experimental error on the muon life time by more than one order of magnitude.

These experimental data have reached such a precision that quantum corrections can be observed. To match this accuracy from the theory side, two-loop radiative corrections to the muon decay in the full electroweak Standard Model are needed. This is a formidable, but not impossible, task. A step in this direction is the calculation of the purely electromagnetic corrections to order α^2 in the Fermi theory, which has been performed by van Ritbergen and Stuart [5,6] (see also [7]). Besides that, also the $\mathcal{O}(N_f \alpha^2)$ corrections in the Standard Model have been considered in [8,9].

It is well known that the decoupling theorem is not applicable to the life time calculation, which means that it is mandatory to also include the contribution of the heavy degrees of freedom. In contrast, the decoupling theorem is fully applicable to the *normalized* electron energy spectrum in muon decay, which has also been measured with an amazing accuracy at the per-mille level. Again the experimental error is expected to be reduced by more than one order of magnitude in a future experiment at TRI-

UMF [4]. The spectrum calculation is different from the one for the life time since the Kinoshita-Lee-Nauenberg theorem [10] is not in effect. Consequently, powers of the large logarithm $\ln(m_\mu/m_e)$ do not cancel in the calculation of the electromagnetic corrections to the spectrum. This becomes obvious when fitting the spectrum corrected to order $\mathcal{O}(\alpha)$ to the Michel spectrum: the resulting effective Michel parameter differs by about 6% from its lowest-order value [11], a correction which is more than 10 times larger than the corresponding correction to the muon life time. At order $\mathcal{O}(\alpha^2)$ the radiative corrections may be expected to be of the order of several per-mille, i.e. they could possibly be visible in present data already, not to speak of future high-precision experiments.

Given this perspective we present in this note the calculation of the hadronic contribution to the energy spectrum of the final-state electron in muon decay. This contribution is not expected to be logarithmically enhanced, but nonetheless is required for an eventual complete second-order calculation. The details of the calculation are given in the following sections.

2. We consider the decay of a muon in its rest system,

$$\mu^-(p) \rightarrow e^-(p') + \nu_\mu(q_1) + \bar{\nu}_e(q_2), \quad (1)$$

and define momenta as shown in (1). The momentum transferred from the charged particles to the neutrino pair, $q = q_1 + q_2$, is then given by

$$q = p - p'. \quad (2)$$

It is convenient to introduce the dimensionless variable

$$x = \frac{2E_e}{m} \quad (3)$$

to denote the ratio of the energy of the decay electron E_e with respect to the muon mass m . Neglecting the electron

^{*} Supported by the Deutsche Forschungsgemeinschaft

^a On leave from Institute for Nuclear Physics, Moscow State University, 119899 Moscow, Russia

^b e-mail: davyd@thep.physik.uni-mainz.de

^c e-mail: hspiesb@thep.physik.uni-mainz.de

mass, $p'^2 = 0$, we see that the kinematically allowed range is

$$0 \leq x \leq 1, \quad (4)$$

and one has

$$q^2 = (1 - x)m^2. \quad (5)$$

The matrix element \mathcal{M} for (1) in the Fermi theory can be calculated most conveniently after a Fierz rearrangement factorizing the amplitude into a current J_μ which describes the μe transition and a current for the $\nu_\mu \nu_e$ interaction. After squaring and summing (averaging) over spins, one can write $|\mathcal{M}|^2$ as a product of two corresponding tensors. The one pertaining to the neutrino interaction can be integrated over the unobserved momenta of the neutrinos independently, leading to

$$N_{\mu\nu} = q_\mu q_\nu - g_{\mu\nu} q^2. \quad (6)$$

This tensor will be contracted with

$$C_{\mu\nu} = J_\mu^* J_\nu, \quad \text{with} \quad J_\mu = \bar{u}_e(p') \Lambda_\mu(q) u_\mu(p), \quad (7)$$

where $\Lambda_\mu(q)$ is the effective vertex of the four-fermion interaction. At the lowest order, $\Lambda_\mu(q)$ is identified with

$$\Lambda_\mu^0 = \frac{G_F}{\sqrt{2}} \gamma_\mu (1 - \gamma_5), \quad (8)$$

where $G_F = (1.16637 \pm 0.00001) \times 10^{-5} \text{ GeV}^{-2}$ is the Fermi coupling constant.

Hadronic contributions to the radiative corrections to the current J_μ at order $\mathcal{O}(\alpha^2)$ are described by the Feynman diagrams shown in Fig. 1. The hadronic vacuum polarization

$$\Pi_{\mu\nu}^{\text{had}}(k^2) = \frac{-i\Pi^{\text{had}}(k^2)}{k^2 + i0} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \quad (9)$$

is inserted in a one-loop vertex correction. The vacuum polarization can be related to the measured cross section for $e^+e^- \rightarrow \text{hadrons}$ with the help of a dispersion relation

$$\Pi^{\text{had}}(k^2) = \frac{\alpha}{3\pi} \int_{s_{\text{thr}}}^{\infty} \frac{ds}{s} R(s) \frac{k^2}{k^2 - s + i0}, \quad (10)$$

where

$$R(s) = \frac{\sigma(s; e^+e^- \rightarrow \text{hadrons})}{\sigma(s; e^+e^- \rightarrow \mu^+\mu^-)} \quad (11)$$

and the integration starts at the two-pion threshold, $s_{\text{thr}} = 4m_\pi^2$. Therefore, the calculation corresponds to a one-loop vertex correction with a photon of mass \sqrt{s} , i.e. using a propagator

$$\frac{-i}{k^2 - s + i0} \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right). \quad (12)$$

The result can be written in the form

$$\Lambda_\mu(q) = \frac{\alpha}{3\pi} \int_{s_{\text{thr}}}^{\infty} \frac{ds}{s} R(s) \tilde{\Lambda}_\mu(s; q, m^2) \quad (13)$$

where the vector function $\tilde{\Lambda}_\mu$ can be decomposed into Lorentz-covariants as

$$\begin{aligned} \tilde{\Lambda}_\mu(s; q, m^2) &= \gamma_\mu \omega_L \left[1 + \tilde{f}(s; q^2, m^2) \right] + \frac{p_\mu + p'_\mu}{m} \omega_R \\ &\times \tilde{g}_+(s; q^2, m^2) + \frac{q_\mu}{m} \omega_R \tilde{g}_-(s; q^2, m^2), \end{aligned} \quad (14)$$

with $\omega_{R,L} = (1 \pm \gamma_5)/2$. The calculation is straightforward and corresponds to that of the one-loop vertex correction in QED, with the difference that (i) the exchanged ‘‘photon’’ is massive with mass \sqrt{s} , (ii) the coupling is purely left-handed, and (iii) the two fermion lines have different masses. In fact, except for small q^2 the electron mass can safely be neglected. Explicit results for the form factors \tilde{f} and \tilde{g}_\pm are given in the appendix. After contraction with the neutrino tensor $N_{\mu\nu}$, (6), only the form factors $\tilde{f}(s; q^2, m^2)$ and $\tilde{g}_+(s; q^2, m^2)$ will remain in the final result.

Performing the dispersion integral one obtains the form factors

$$\begin{aligned} f(q^2, m^2) &= \frac{\alpha}{3\pi} \int_{s_{\text{thr}}}^{\infty} \frac{ds}{s} R(s) \tilde{f}(s; q^2, m^2), \\ g_+(q^2, m^2) &= \frac{\alpha}{3\pi} \int_{s_{\text{thr}}}^{\infty} \frac{ds}{s} R(s) \tilde{g}_+(s; q^2, m^2). \end{aligned} \quad (15)$$

Since we are interested in the $\mathcal{O}(\alpha^2)$ correction, it is sufficient to keep only terms of first order in f and g_+ in the decay spectrum which then can be written in the form

$$\begin{aligned} \frac{1}{\Gamma_0} \frac{d\Gamma}{dx} &= 2x^2 [(3 - 2x)(1 + 2f(x)) + xg_+(x)] \\ &= \frac{1}{\Gamma_0} \frac{d\Gamma}{dx} \Big|_{\text{Born}} (1 + r(x)), \end{aligned} \quad (16)$$

with

$$\Gamma_0 = \frac{G_F^2 m^5}{192\pi^3}, \quad \frac{d\Gamma}{dx} \Big|_{\text{Born}} = 2x^2(3 - 2x)\Gamma_0, \quad (17)$$

and

$$r(x) = 2f(x) + \frac{x}{3 - 2x} g_+(x), \quad (18)$$

where $f(x) \equiv f((1 - x)m^2, m^2)$ and $g_+(x) \equiv g_+((1 - x)m^2, m^2)$.

3. For our purpose, the function $R(s)$ describing the hadronic cross section of e^+e^- annihilation can be modeled by a combination of experimental data and analytical results from perturbative QCD. Since we are going to calculate a small correction, it is not necessary to invoke the most sophisticated treatment as needed, for example, when calculating the hadronic contribution to the fine structure constant $\alpha(m_Z)$. At low $s < 2.5 \text{ GeV}^2$ we use experimental data from ALEPH parametrized in [12] or provided directly by ALEPH [13] from a measurement of the isovector τ spectral function. These data are complemented by the resonance contributions from the isospin-0 light mesons ω and ϕ . Above $s = 2.5 \text{ GeV}^2$ we use

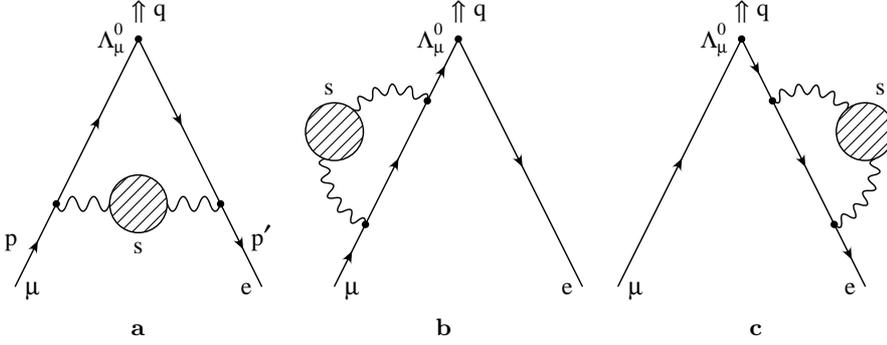


Fig. 1a–c. Feynman diagrams describing a self energy insertion in the photonic one-loop corrections to the μe vertex

the QCD prediction for $R(s)$ due to light quarks at order $\mathcal{O}(\alpha_s)$.

Since the data in the $c\bar{c}$ -channel published by various groups are in a large part of the energy range inconsistent, we apply in this case the QCD-based approach of analytic continuation by duality [14]. The data region can be chosen to extend only over the sub-threshold resonances, i.e. one can calculate the contribution coming from the $c\bar{c}$ -channel by a combination of data describing the $J/\Psi(1S)$ and $J/\Psi(2S)$ resonances and the prediction of perturbative QCD. We checked that the results obtained this way are consistent with those of the standard approach using the new BES data [15].

The correction to the total decay rate,

$$\Delta\Gamma = \int_0^1 dx \frac{d\Gamma}{dx}, \quad (19)$$

was calculated before in [5]. Our result,

$$\Delta\Gamma_{\text{had}} \simeq -0.0421 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \quad (20)$$

agrees perfectly with the corresponding number -0.042 given in [5]. The resulting corrections to the spectrum (16) are shown in Fig. 2. At small x , the corrections are positive, but the correction to the total decay width is dominated by the negative values at $x \gtrsim 0.18$. The dependence of the form factors on x is to a very good approximation linear:

$$\begin{aligned} f(x) &\simeq (0.0071 - 0.0378x) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \\ g_+(x) &\simeq -0.0067 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \\ r(x) &\simeq (0.0148 - 0.0813x) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0. \end{aligned} \quad (21)$$

For g_+ , the coefficient of the term linear in x is very small and is therefore omitted. Note that this behaviour cannot be described by a simple redefinition of the Michel parameter. Since G_F is a free parameter in the Fermi theory, the correction to the total decay width is not observable; it can be absorbed by a suitable redefinition of the Fermi constant. However, the modification of the spectrum is, in principle, observable.

The correction to the total decay rate can be split up into the various contributions to the hadronic vacuum polarization, as shown in Table 1. The form factor f of the

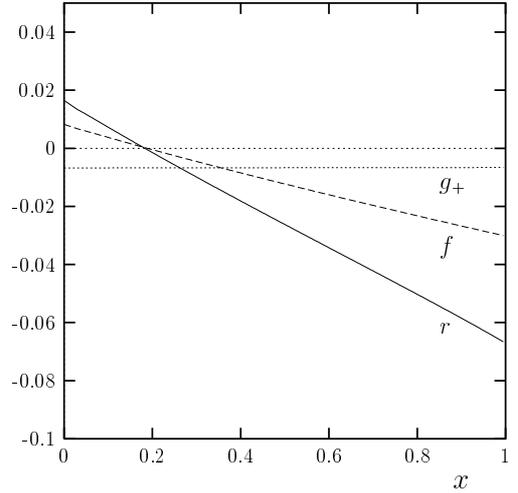


Fig. 2. Results for the form factors f , g_+ and r defined in (15), (18)

Table 1. Contributions to the corrections of the total decay rate

Contributions to $\Delta\Gamma_{\text{had}}$			
1	$0 < s < 0.2 \text{ GeV}^2$	-0.00129	3.1%
2	ω	-0.00223	5.3%
3	ϕ	-0.00264	6.3%
4	$0.2 < s < 2.5 \text{ GeV}^2$	-0.02804	66.6%
5	$s > 2.5 \text{ GeV}^2$	-0.00564	13.4%
6	$J/\Psi(1S)$	-0.00066	1.6%
7	$J/\Psi(2S)$	-0.00017	0.4%
8	charm, $s > 4m_c^2$	-0.00138	3.3%
9	bottom, $s > 4m_b^2$	-0.00003	0.1%
Sum		-0.04207	100%

γ_μ term contributes -0.0387 , whereas the correction due to g_+ , -0.0033 , is smaller by one order of magnitude. The total correction (the contributions due to f and g_+) is saturated to 81% (80.8% and 89%, respectively) by the contributions from small s below 2.5 GeV^2 . Only 5% of $\Delta\Gamma_{\text{had}}$ is due to charmed states, and the bottom sector is completely negligible.

The numerical results given here take into account the $\mathcal{O}(\alpha_s)$ QCD corrections in $R(s)$. Using the leading-

order expression for $R(s)$, i.e. without the correction factor $(1 + \alpha_s/\pi)$ in the light-quark contribution at large s and the charm-quark contribution, the final result would be -0.04149 , i.e. changed by 1.4%. We conclude that a more refined treatment which would include higher orders of perturbative QCD and mass-dependent corrections in the heavy-quark sector is not required for our purpose.

The evaluation of the dispersion integrals has been performed using standard numerical integration routines up to a value s_{\max} of several hundred GeV^2 . The contribution above this value was obtained with the help of `Maple` using the asymptotic expansion of the form factors for large s (see (35)–(37) in the Appendix). For intermediate values of s , good consistency of both procedures has been verified.

4. The same set of formulae can be used to calculate the contributions from a $\mu^+\mu^-$ or a $\tau^+\tau^-$ loop insertion. We find that the tau loop gives a very small contribution, about 1.5% of the one from the muon loop, in agreement with [5]. Therefore we give only results for the muon loop, where one has to insert in (15)

$$R(s) \rightarrow \left(1 + \frac{2m^2}{s}\right) \sqrt{1 - \frac{4m^2}{s}} \quad \text{and} \quad s_{\text{thr}} \rightarrow 4m^2. \quad (22)$$

With this input we obtain

$$\Delta\Gamma_{\text{muon}} \simeq -0.0364 \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \quad (23)$$

which perfectly agrees with the exact result given in [5]. The results of a linear fit of the form factors for the muon-loop insertion are:

$$\begin{aligned} f_{\text{muon}}(x) &\simeq (0.0130 - 0.0414x) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \\ g_{+, \text{muon}}(x) &\simeq (-0.0090 + 0.0005x) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0, \\ r_{\text{muon}}(x) &\simeq (0.0267 - 0.0898x) \left(\frac{\alpha}{\pi}\right)^2 \Gamma_0. \end{aligned} \quad (24)$$

5. To summarize, we found that the energy spectrum in the decay of an unpolarized muon is corrected by a smooth function in x due to hadronic contributions at order $\mathcal{O}(\alpha^2)$. At both ends of the spectrum no particularly outstanding enhancement or suppression is observed.

The calculations described in this paper constitute only the most straightforward part of a full calculation which would be necessary before the expected future high-precision data can be confronted with theoretical predictions. This will not only be necessary for a meaningful test of the electroweak Standard Model, but also when searching for physics beyond the Standard Model [16].

Appendix

The form factors introduced in (14) are given explicitly in the following:

$$\tilde{f}(s; q^2, m^2)$$

$$\begin{aligned} &= -\frac{\alpha}{4\pi} \left\{ 2(m^2 - q^2 - s) \left[1 + \frac{q^2 s}{(m^2 - q^2)^2} \right] C_0 \right. \\ &+ \left[\frac{2q^2 s}{(m^2 - q^2)^2} - \frac{2q^2}{(m^2 - q^2)} + 1 \right] B_0(q^2; m^2, 0) \\ &+ \left[\frac{2q^2}{m^2 - q^2} - \frac{s(m^2 + q^2)}{(m^2 - q^2)^2} \right] B_0(m^2; m^2, s) \\ &\left. + \left(\frac{1}{s} - \frac{1}{m^2 - q^2} \right) A(s) + 2 \right\} + \tilde{f}_{\text{SE}}(s; q^2, m^2), \quad (25) \end{aligned}$$

$$\tilde{g}_+(s; q^2, m^2)$$

$$\begin{aligned} &= -\frac{\alpha}{4\pi} \frac{m^2}{q^2} \left\{ \frac{2q^2 s}{m^2 - q^2} \left[\frac{3q^2 s}{(m^2 - q^2)^2} + 2 \right] C_0 \right. \\ &- \frac{q^2}{m^2 - q^2} \left[\frac{6q^2 s}{(m^2 - q^2)^2} + 1 \right] B_0(q^2; m^2, 0) - \frac{q^2}{m^2 - q^2} \\ &+ \frac{q^2}{m^2 - q^2} \left[\frac{6m^2 s}{(m^2 - q^2)^2} - \frac{s(4m^2 - q^2)}{m^2(m^2 - q^2)} + 2 \right] \\ &\times B_0(m^2; m^2, s) + \frac{q^2}{m^2 - q^2} \left[\frac{3}{m^2 - q^2} - \frac{1}{m^2} \right] A(s) \\ &\left. + \left[\frac{1}{m^2 - q^2} - \frac{1}{m^2} \right] A(m^2) \right\}, \quad (26) \end{aligned}$$

$$\tilde{g}_-(s; q^2, m^2)$$

$$\begin{aligned} &= \frac{\alpha}{4\pi} \frac{m^2}{q^2} \left\{ \frac{2q^2 s}{m^2 - q^2} \left[\frac{3q^2 s}{(m^2 - q^2)^2} + \frac{2s}{m^2 - q^2} + 2 \right] C_0 \right. \\ &- \frac{q^2}{m^2 - q^2} + \frac{1}{m^2 - q^2} \left[-\frac{6m^2 q^2 s}{(m^2 - q^2)^2} + \frac{2q^2 s}{m^2 - q^2} \right. \\ &+ 2m^2 - 3q^2 \left. \right] B_0(q^2; m^2, 0) + \frac{q^2}{m^2 - q^2} \left[\frac{6m^2 s}{(m^2 - q^2)^2} \right. \\ &+ \frac{s}{m^2 - q^2} + \frac{s}{m^2} + 2 \left. \right] B_0(m^2; m^2, s) + \frac{q^2}{m^2 - q^2} \\ &\times \left(\frac{1}{m^2} + \frac{3}{m^2 - q^2} \right) A(s) + \left(\frac{1}{m^2} + \frac{1}{m^2 - q^2} \right) A(m^2) \left. \right\}, \quad (27) \end{aligned}$$

where $C_0 = C_0(m^2, 0, q^2; m^2, s, 0)$ is the three-point integral (cf. Fig. 1a), defined in (34) below. B_0 and A denote the tadpole and two-point integrals [17, 18], respectively (see (30)–(32) below).

Self-energy diagrams (cf. Fig. 1b, c) contribute to the coefficient of γ_μ only and are given by

$$\begin{aligned} &\tilde{f}_{\text{SE}}(s; q^2, m^2) \\ &= \frac{\alpha}{8\pi} \left\{ 2(s + 2m^2) \frac{\partial}{\partial p^2} B_0(p^2; m^2, s) \Big|_{p^2=m^2} \right. \\ &\left. - \frac{s}{m^2} B_0(m^2; m^2, s) - \frac{s + m^2}{sm^2} A(s) \right\} \end{aligned}$$

$$+\frac{1}{m^2}A(m^2) + 1 + \frac{3}{2}\left. \right\}, \quad (28)$$

where the last term, $\frac{3}{2}$, comes from the self energy on the massless (electron) leg. Using recurrence relations [19] (see also Appendix A of [20]), the derivative in (28) can be represented as

$$\begin{aligned} & \left. \frac{\partial}{\partial p^2} B_0(p^2; m^2, s) \right|_{p^2=m^2} \\ &= \frac{1}{m^2(s-4m^2)} \left[-(s-3m^2)B_0(m^2; m^2, s) \right. \\ & \quad \left. + A(m^2) - \left(1 - \frac{2m^2}{s}\right) A(s) + m^2 \right]. \end{aligned} \quad (29)$$

The required tadpole and two-point integrals are [17,18]

$$A(m^2) = m^2 \left[-\Delta - 1 + \ln \frac{m^2}{\mu_{\text{DR}}^2} \right], \quad (30)$$

$$\begin{aligned} B_0(m^2; m^2, s) &= \Delta + 2 - \ln \frac{m^2}{\mu_{\text{DR}}^2} - \frac{s}{2m^2} \ln \frac{s}{m^2} \\ & \quad + \frac{s}{2m^2} \beta_s \ln \left(\frac{1+\beta_s}{1-\beta_s} \right), \end{aligned} \quad (31)$$

$$\begin{aligned} B_0(q^2; m^2, 0) &= \Delta + 2 - \ln \frac{m^2}{\mu_{\text{DR}}^2} + \frac{m^2 - q^2}{q^2} \\ & \quad \times \ln \left(\frac{m^2 - q^2}{m^2} \right), \end{aligned} \quad (32)$$

where

$$\beta_s \equiv \sqrt{1 - \frac{4m^2}{s}}, \quad (33)$$

and μ_{DR} is the scale parameter of dimensional regularization. In (30)–(32), terms containing $\Delta = 1/\varepsilon - \ln \pi - \gamma_E$ represent the ultraviolet singularities which cancel in the final results (25)–(27).

Finally, we need the three-point scalar function C_0 [17] for positive values of q^2 . In this case it can be written in the following form:

$$\begin{aligned} & C_0(m^2, 0, q^2; m^2, s, 0) \\ &= \frac{1}{m^2 - q^2} \left\{ \text{Li}_2 \left(1 - \frac{m^2}{q^2} \right) + \text{Li}_2 \left(-\frac{(m^2 - q^2)^2}{sq^2} \right) \right. \\ & \quad - \text{Li}_2 \left[\frac{1}{2} \left(1 - \frac{m^2}{q^2} \right) (1 + \beta_s) \right] - \text{Li}_2 \left[\frac{1}{2} \left(1 - \frac{m^2}{q^2} \right) (1 - \beta_s) \right] \\ & \quad + \frac{1}{2} \ln \left[\frac{(m^2 - q^2)^2}{sm^2} \right] \ln \left[1 + \frac{(m^2 - q^2)^2}{sq^2} \right] - \frac{1}{2} \ln \frac{m^2}{q^2} \\ & \quad \left. \times \ln \frac{m^2}{s} + \frac{1}{2} \ln \left[\frac{m^2 + q^2 + (q^2 - m^2)\beta_s}{m^2 + q^2 - (q^2 - m^2)\beta_s} \right] \ln \left(\frac{1 + \beta_s}{1 - \beta_s} \right) \right\}. \end{aligned} \quad (34)$$

For convenience, we also give asymptotic expansions of the form factors (25)–(27) valid for large s ,

$$\begin{aligned} \frac{4\pi}{\alpha} \tilde{f} &= -\frac{1}{3s(q^2)^2} \left\{ (m^2 - q^2)^2 (m^2 + 2q^2) \ln \left(\frac{m^2 - q^2}{m^2} \right) \right. \\ & \quad + (q^2)^2 (3m^2 - 2q^2) \ln \frac{s}{m^2} + q^2 \left[m^4 - \frac{9}{2} m^2 q^2 \right. \\ & \quad \left. \left. - \frac{11}{3} (q^2)^2 \right] \right\} + \frac{1}{6s^2(q^2)^3} \left\{ (m^2 - q^2)^4 (m^2 + q^2) \right. \\ & \quad \times \ln \left(\frac{m^2 - q^2}{m^2} \right) - (q^2)^3 [35m^4 - 3m^2 q^2 + (q^2)^2] \\ & \quad \times \ln \frac{s}{m^2} + q^2 \left[m^8 - \frac{5}{2} m^6 q^2 + \frac{697}{12} m^4 (q^2)^2 \right. \\ & \quad \left. \left. + \frac{11}{4} m^2 (q^2)^3 - \frac{13}{12} (q^2)^4 \right] \right\} + \mathcal{O}(s^{-3}), \end{aligned} \quad (35)$$

$$\begin{aligned} \frac{4\pi}{\alpha} \tilde{g}_+ &= -\frac{m^2}{3s} + \frac{m^2}{6s^2(q^2)^3} \left\{ (m^2 - q^2)^4 \ln \left(\frac{m^2 - q^2}{m^2} \right) \right. \\ & \quad + (q^2)^3 (4m^2 - q^2) \ln \frac{s}{m^2} + q^2 \left[m^6 - \frac{7}{2} m^4 q^2 \right. \\ & \quad \left. \left. - \frac{8}{3} m^2 (q^2)^2 + \frac{5}{12} (q^2)^3 \right] \right\} + \mathcal{O}(s^{-3}), \end{aligned} \quad (36)$$

$$\begin{aligned} \frac{4\pi}{\alpha} \tilde{g}_- &= \frac{4m^2}{3s(q^2)^3} \left\{ (m^2 - q^2)^3 \ln \left(\frac{m^2 - q^2}{m^2} \right) + (q^2)^3 \ln \frac{s}{m^2} \right. \\ & \quad + q^2 \left[m^4 - \frac{5}{2} m^2 q^2 + \frac{7}{12} (q^2)^2 \right] \left. \right\} - \frac{m^2}{6s^2(q^2)^4} \\ & \quad \times \left\{ (m^2 - q^2)^4 (6m^2 - 5q^2) \ln \left(\frac{m^2 - q^2}{m^2} \right) \right. \\ & \quad - (q^2)^4 (26m^2 - 5q^2) \ln \frac{s}{m^2} + q^2 \left[6m^8 - 26m^6 q^2 \right. \\ & \quad \left. \left. + \frac{87}{2} m^4 (q^2)^2 - \frac{25}{6} m^2 (q^2)^3 + \frac{23}{12} (q^2)^4 \right] \right\} \\ & \quad + \mathcal{O}(s^{-3}). \end{aligned} \quad (37)$$

These expressions turned out to be useful for the numerical evaluation of the dispersion integrals in the large- s region.

The asymptotic values for $x = 0$ are as follows:

$$\begin{aligned} \frac{4\pi}{\alpha} \tilde{f} \Big|_{x=0} &= -\frac{1}{12m^4} \left\{ (s+2m^2)(7s-22m^2) \frac{1}{\beta_s} \ln \left(\frac{1+\beta_s}{1-\beta_s} \right) \right. \\ & \quad + (18m^4 - 6m^2 s - 7s^2) \ln \frac{s}{m^2} \\ & \quad \left. + m^2 (33m^2 + 14s) \right\}, \end{aligned}$$

$$\frac{4\pi}{\alpha} \tilde{g}_+ \Big|_{x=0} = \frac{1}{4m^4} \left\{ s(2m^2 - s)\beta_s \ln \left(\frac{1+\beta_s}{1-\beta_s} \right) + (2m^4 - 4m^2s + s^2) \ln \frac{s}{m^2} + m^2(5m^2 - 2s) \right\},$$

$$\frac{4\pi}{\alpha} \tilde{g}_- \Big|_{x=0} = \frac{1}{12m^4} \left\{ s(2m^2 - 5s)\beta_s \ln \left(\frac{1+\beta_s}{1-\beta_s} \right) - (6m^4 + 12m^2s - 5s^2) \ln \frac{s}{m^2} + m^2(9m^2 - 10s) \right\}.$$

Note that there are no contributions of the order $\mathcal{O}(x \ln x)$.

References

1. D. E. Groom et al. (Particle Data Group), *Eur. Phys. J.* **C15** (2000) 1
2. R. M. Carey et al., “Proposal for: A Precision Measurement of the Positive Muon Lifetime Using a Pulsed Muon Beam and the μLAN Detector”, May 25, 1999, PSI, (Villigen 1999)
3. S. N. Nakamura, “Precise measurement of μ^+ lifetime and test of exponential decay law”, RIKEN-R77 (1998)
4. R. Abegg et al., “Precision Measurement of the Michel Parameters of μ^+ Decay”, TRIUMF Research Proposal E614, 1997
5. T. van Ritbergen, R. G. Stuart, *Phys. Lett.* **B437** (1998) 201
6. T. van Ritbergen, R. G. Stuart, *Phys. Rev. Lett.* **82** (1999) 488; *Nucl. Phys.* **B564** (2000) 343
7. M. Steinhauser, T. Seidensticker, *Phys. Lett.* **B467** (1999) 271
8. P. Malde, R. G. Stuart, *Nucl. Phys.* **B552** (1999) 41
9. A. Freitas, S. Heinemeyer, W. Hollik, W. Walter, G. Weiglein, *Phys. Lett.* **B495** (2000) 338; *Nucl. Phys. B (Proc. Suppl.)* 89 (2000) 82 (hep-ph/0007129)
10. T. Kinoshita, *J. Math. Phys.* **3** (1962) 650; T. D. Lee, M. Nauenberg, *Phys. Rev.* **133** (1964) B1549
11. R. E. Behrends, R. J. Finkelstein, A. Sirlin, *Phys. Rev.* **101** (1956) 866
12. M. Davier, L. Girlanda, A. Höcker, *J. Stern, Phys. Rev.* **D58** (1998) 096014
13. R. Barate et al. (ALEPH Collaboration), *Eur. Phys. J.* **C4** (1998) 409
14. M. Kremer, N. F. Nasrallah, N. A. Papadopoulos, K. Schilcher, *Phys. Rev.* **D34** (1986) 2127
15. J. Z. Bai et al. (BES Collaboration), *Phys. Rev. Lett.* **84** (2000) 594; Y. Shen, Proc. IV International Conference on Hyperons, Charm and Beauty Hadrons, June 27–30 2000, Valencia, Spain. Z. Zhao, Talk 05a-07 in Proc. XXXth International Conference on High Energy Physics, ICHEP2000, July 27 – August 2, 2000, Osaka, Japan
16. F. Scheck, *Phys. Rep.* **44** (1978) 187; W. Fetscher and H.-J. Gerber, in: “Precision Tests of the Standard Electroweak Model”, P. Langacker (ed.), *Advanced Series on Directions in High Energy Physics – Vol. 14*, p. 657 (World Scientific, Singapore, 1995); Y. Kuno, Y. Okada, *Rev. Mod. Phys.* **73** (2001) 151
17. G. 't Hooft, M. Veltman, *Nucl. Phys.* **B153** (1979) 365
18. G. Passarino, M. Veltman, *Nucl. Phys.* **B160** (1979) 151
19. F. V. Tkachov, *Phys. Lett.* **100B** (1981) 65; K. G. Chetyrkin, F. V. Tkachov, *Nucl. Phys.* **B192** (1981) 159
20. F. A. Berends, A. I. Davydychev, V. A. Smirnov, *Nucl. Phys.* **B478** (1996) 59