Consider two populations of longitudinally polarized muons at rest in the laboratory, one fully polarized along Oz ($N_1(t)$ at time t) and the other fully polarized in the opposite direction ($N_2(t)$ at time t). They both decay with the muon lifetime (decay rate λ_0), but, due to a spin-flip interaction, there is a transfer from one population to the other (transfer rate λ). This gives the coupled differential equations:

$$\begin{aligned} \frac{dN_1}{dt} &= -\lambda_0 N_1 - \lambda N_1 + \lambda N_2 \\ \frac{dN_2}{dt} &= -\lambda_0 N_2 - \lambda N_2 + \lambda N_1 \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{d}{dt}(N_1 + N_2) &= -\lambda_0 (N_1 + N_2) \\ \frac{d}{dt}(N_1 - N_2) &= -(\lambda_0 + 2\lambda)(N_1 - N_2) \end{aligned} \tag{2}$$

The solution, including initial conditions, is:

$$\begin{aligned} N_1(t) + N_2(t) &= [N_1(0) + N_2(0)] \exp(-\lambda_0 t) \\ N_1(t) - N_2(t) &= [N_1(0) - N_2(0)] \exp[-(\lambda_0 + 2\lambda)t] \end{aligned} \tag{3}$$

and:

$$\frac{N_1(t) - N_2(t)}{N_1(t) + N_2(t)} = \frac{N_1(0) - N_2(0)}{N_1(0) - N_2(0)} \exp(-2\lambda t) \tag{4}$$

This is the degree of polarization $P(t)$ of the muon ensemble:

$$P(t) = P(0) \exp(-2\lambda t) \tag{5}$$

It decays with the decay constant 2λ.
Also:

$$\begin{aligned} N_1(t) &= \frac{1}{2} \exp(-\lambda_0 t) \left\{ N_1(0)[1 + \exp(-2\lambda t)] + N_2(0)[1 - \exp(-2\lambda t)] \right\} \\ N_2(t) &= \frac{1}{2} \exp(-\lambda_0 t) \left\{ N_1(0)[1 - \exp(-2\lambda t)] + N_2(0)[1 + \exp(-2\lambda t)] \right\} \end{aligned} \tag{6}$$

Assuming that the initial state is a pure state of polarization along Oz, the initial conditions are:

$$N_1(0) = 1 \qquad N_2(0) = 0 \tag{7}$$

this gives:

$$\begin{aligned} N_1(t) &= \frac{1}{2} \exp(-\lambda_0 t)[1 + \exp(-2\lambda t)] \\ N_2(t) &= \frac{1}{2} \exp(-\lambda_0 t)[1 - \exp(-2\lambda t)] \end{aligned} \tag{8}$$

or, by using the dimensionless variable τ:

$$
\begin{aligned}
N_1(t) &= \frac{1}{2}\exp(-\tau)[1+\exp(-2\alpha\tau)] \\
N_2(t) &= \frac{1}{2}\exp(-\tau)[1-\exp(-2\alpha\tau)]
\end{aligned} \tag{9}
$$

with $\lambda_0 t = \tau$ and $\alpha = \lambda/\lambda_0$.

In the following figures are plotted, for three different values of α:

1. $N_1(t)$: first population of muons.
2. $N_2(t)$: second population of muons.
3. $N_1(t) + N_2(t)$: total population of muons. It decays with the muon decay constant λ_0.
4. $P(t) = (N_1(t) - N_2(t))/(N_1(t) + N_2(t))$: degree of polarization. It decays with the decay constant 2λ.

The time τ is in units of the muon lifetime.

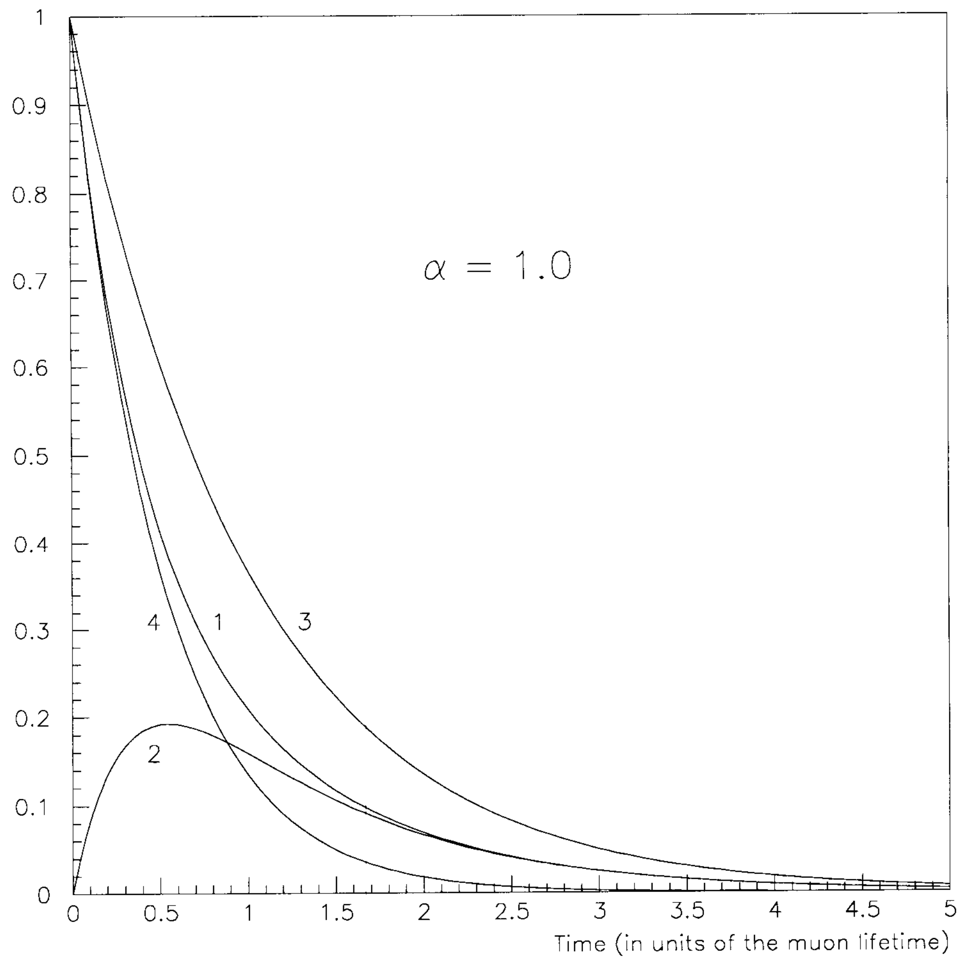

Figure 1: Time evolution of the two populations of muons. Case $\alpha = 1.0$.

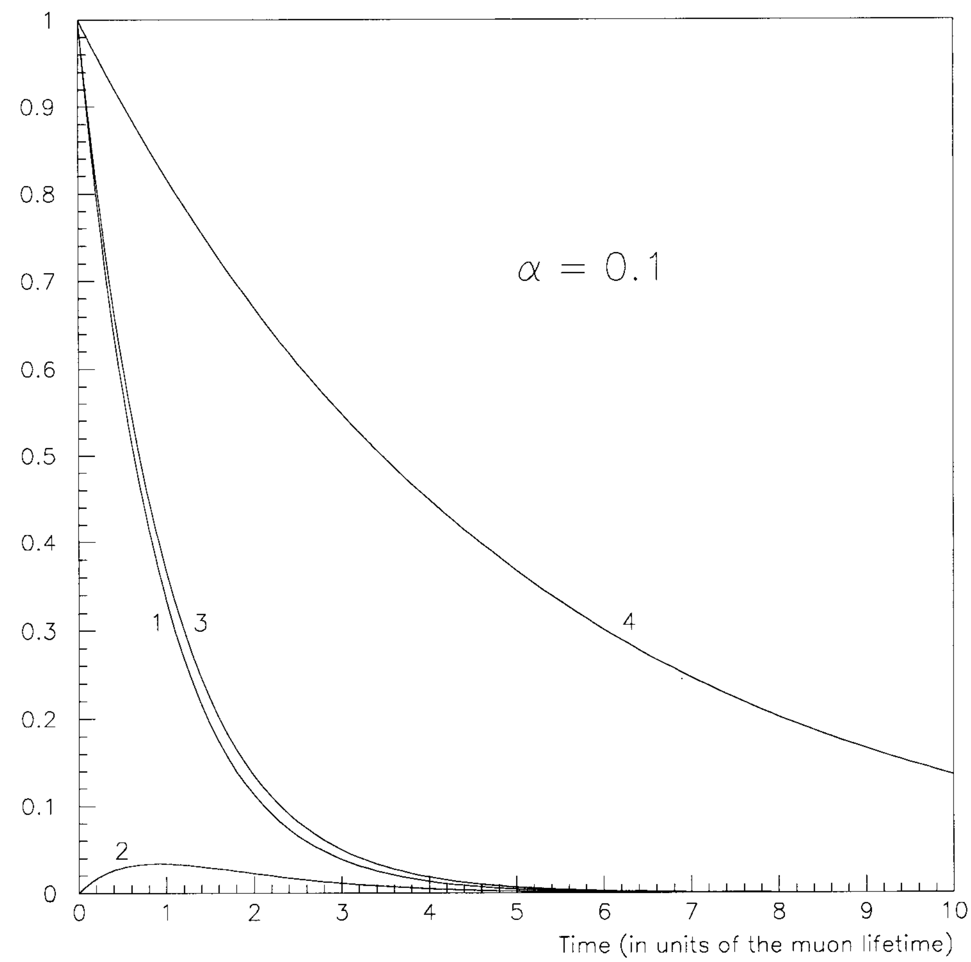

Figure 2: Time evolution of the two populations of muons. Case $\alpha = 0.1$.

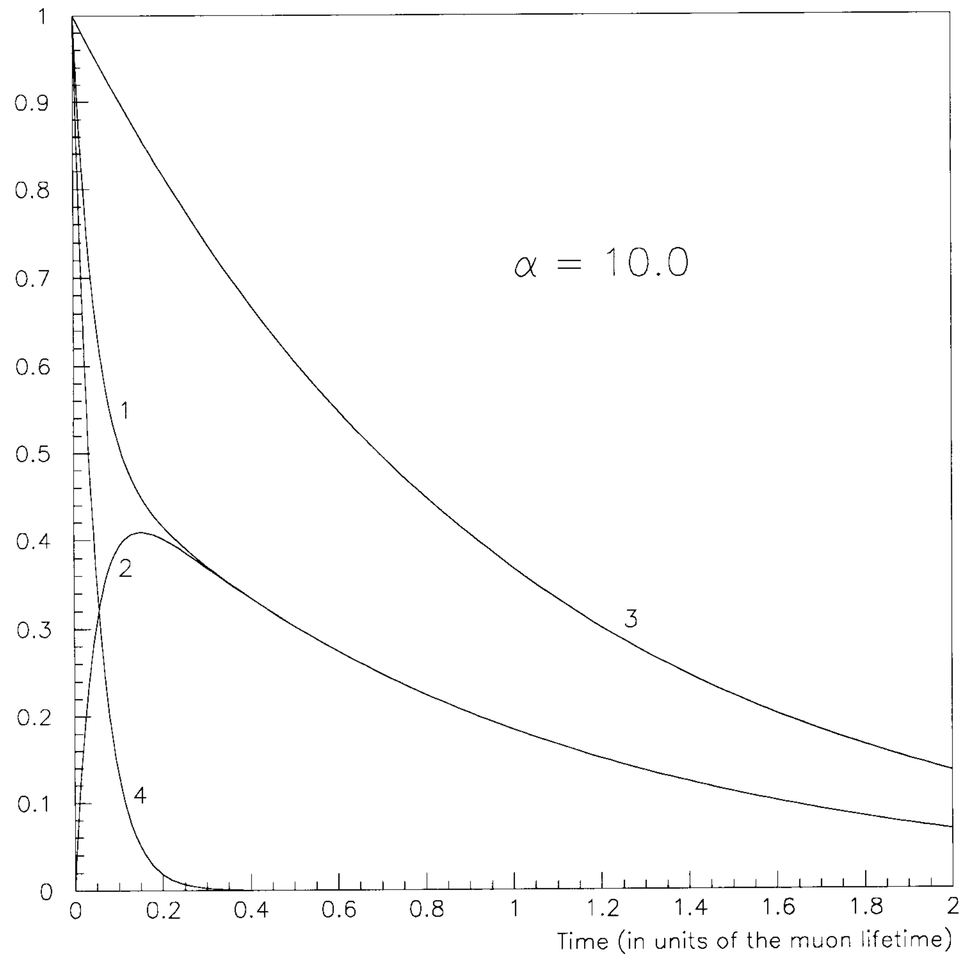

Figure 3: Time evolution of the two populations of muons. Case $\alpha = 10.0$.

Let us now study the shape of the positron spectrum and the positron asymmetry as a function of time. The general expression for the energy and angular distributions of the positron is:

$$\frac{d^2\Gamma}{dx\,d(\cos\theta)} = (x^2 - x_0^2)^{1/2}\Big\{ 6x(1-x) + \frac{4}{3}\,\rho(4x^2 - 3x - x_0^2) + 6\eta x_0(1-x)$$

$$+P_\mu\xi\cos\theta\;(x^2 - x_0^2)^{1/2}\Big[2(1-x) + \frac{4}{3}\,\delta\Big(4x - 4 + (1-x_0^2)^{1/2}\Big)\Big]\Big\} \tag{10}$$

In the following the positron mass will be neglected ($x_0 = 0$):

$$\frac{d^2\Gamma}{dx\,d(\cos\theta)} = x^2\Big\{6(1-x) + \frac{4}{3}\,\rho(4x-3) + P_\mu\xi\cos\theta\Big[2(1-x) + \frac{4}{3}\,\delta\Big(4x-3\Big)\Big]\Big\} \tag{11}$$

Assuming the standard values for ρ and δ:

$$\frac{d^2\Gamma}{dx\,d(\cos\theta)} = x^2\Big[(3-2x) + P_\mu\xi\cos\theta(2x-1)\Big] \tag{12}$$

The asymmetry is:

$$A(x) = P_\mu\xi\cos\theta\frac{2x-1}{3-2x} \tag{13}$$

For $P_\mu\xi = +1$:

$$\frac{d^2\Gamma}{dx\,d(\cos\theta)} = x^2\Big[(3-2x) + \cos\theta\;(2x-1)\Big] \tag{14}$$

$$A(x) = \cos\theta\frac{2x-1}{3-2x} \tag{15}$$

For $\cos\theta = +1$:

$$\frac{d\Gamma}{dx} = 2x^2 \tag{16}$$

$$A(x) = \frac{2x-1}{3-2x} \tag{17}$$

For $\cos\theta = -1$:

$$\frac{d\Gamma}{dx} = 4x^2(1-x) \tag{18}$$

$$A(x) = -\frac{2x-1}{3-2x} \tag{19}$$

The positron spectra for three values of $\cos\theta$ are shown on Figure 4.

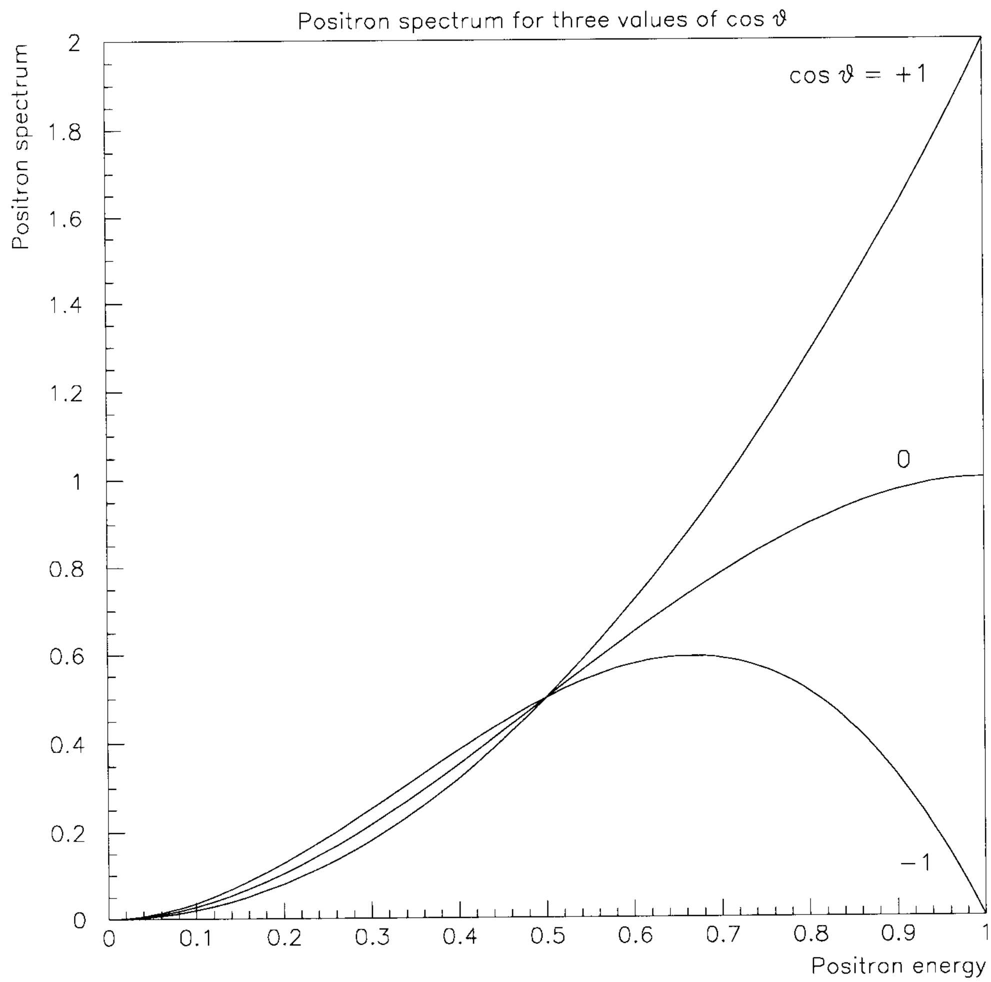

Figure 4: Positron spectrum for three values of $\cos\theta$.

The positron asymmetries for three values of $\cos\theta$ are shown on Figure 5.

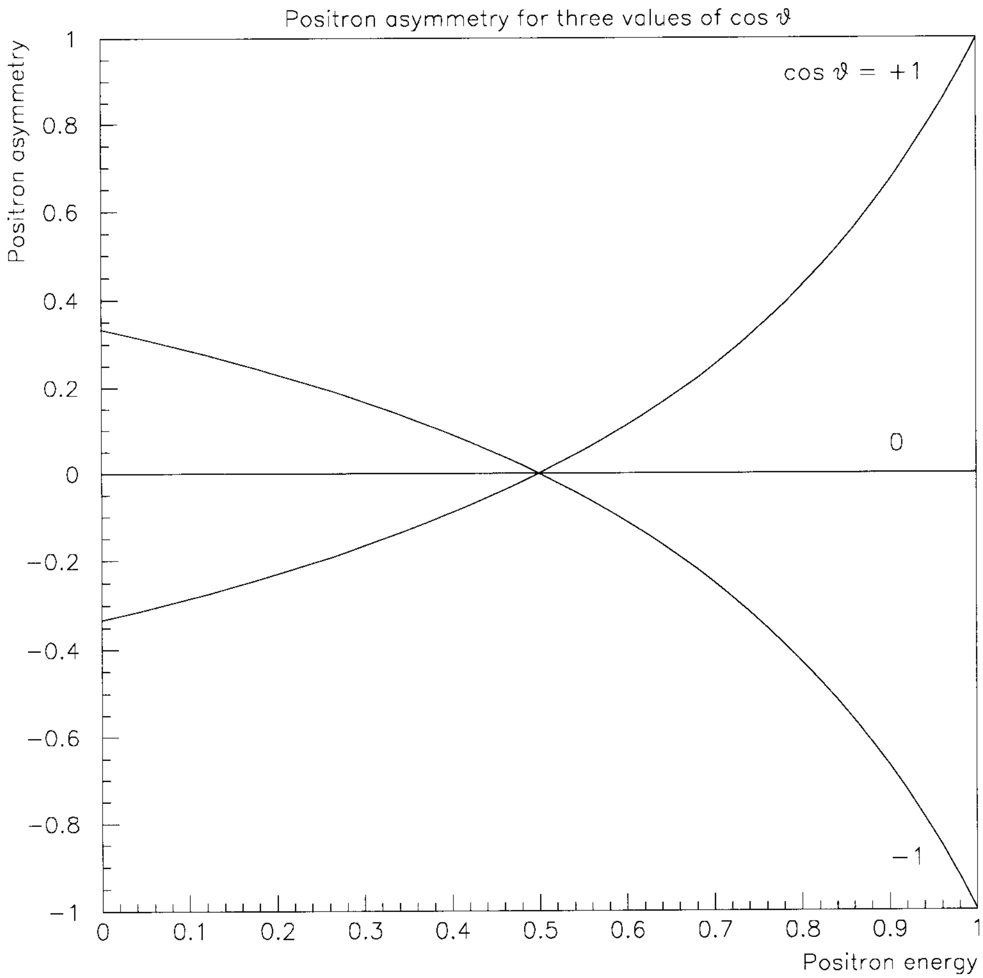

Figure 5: Positron asymmetry for three values of $\cos\theta$.

Assuming that positrons are detected in one direction, the positron spectrum will be a function of time:

$$S(x,t) = N_1(t)[2x^2] + N_2(t)[4x^2(1-x)] \tag{20}$$

and the asymmetry:

$$A(x,t) = \frac{N_1(t) - N_2(t)}{N_1(t) + N_2(t)} \frac{2x-1}{3-2x} \tag{21}$$

The following figures show the content of various energy bins (1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0,1) for two cases of initial conditions:

- $N_1 = 1$, $N_2 = 0$.
- $N_1 = 0$, $N_2 = 1$.

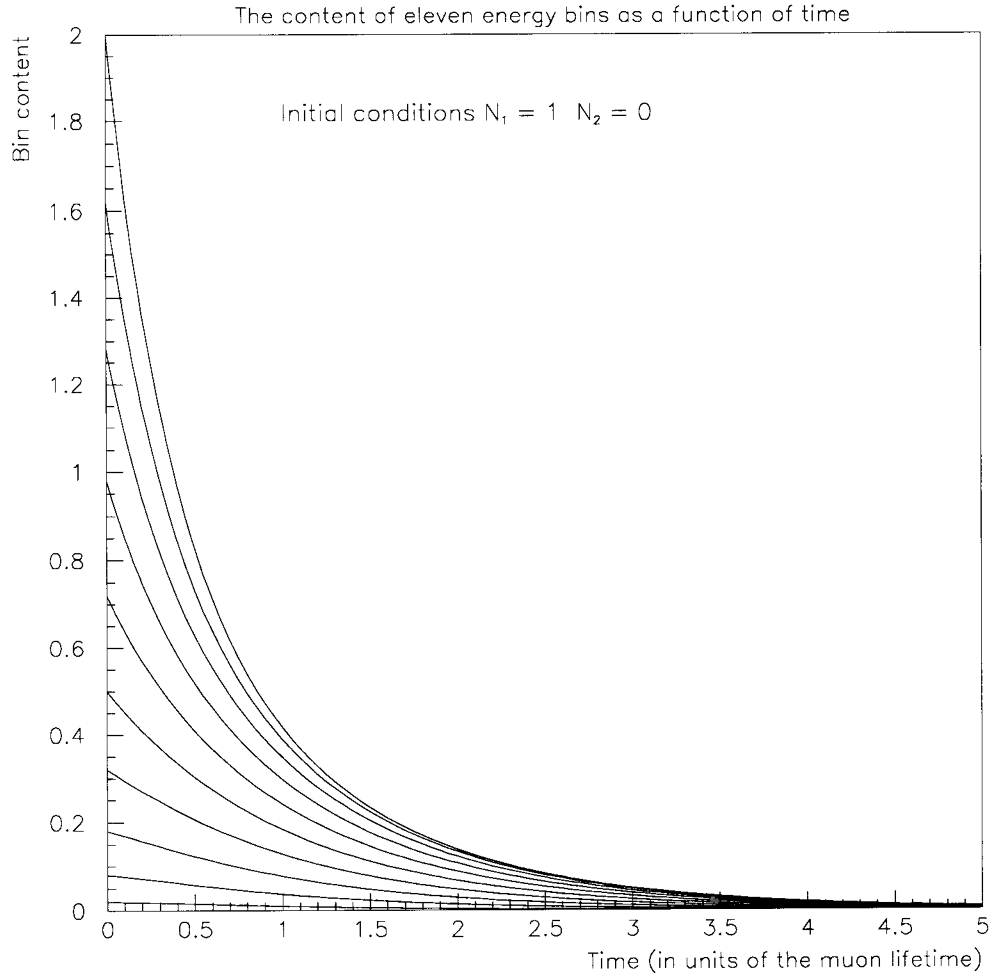

Figure 6: Content of eleven energy bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

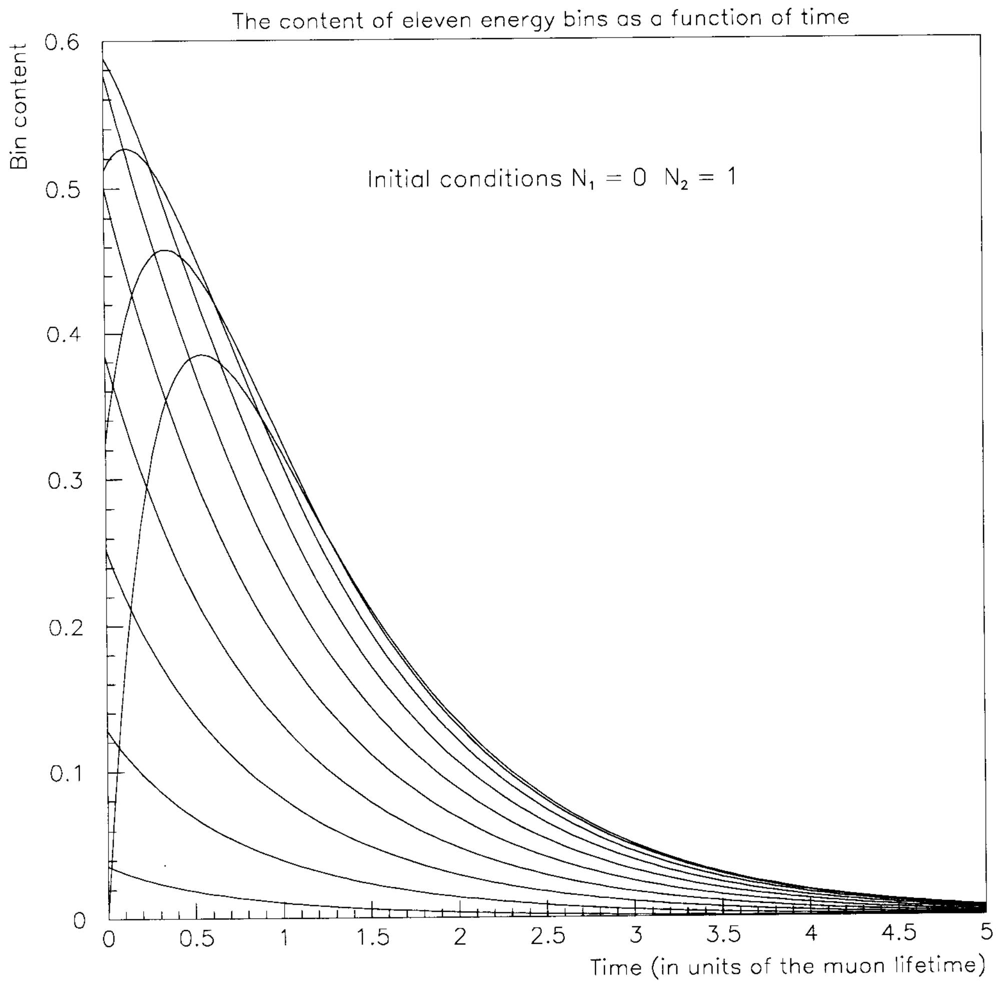

Figure 7: Content of eleven energy bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

The following figures show the content of various asymmetry bins (1.0, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0,1) for two cases of initial conditions:

- $N_1 = 1$, $N_2 = 0$.
- $N_1 = 0$, $N_2 = 1$.

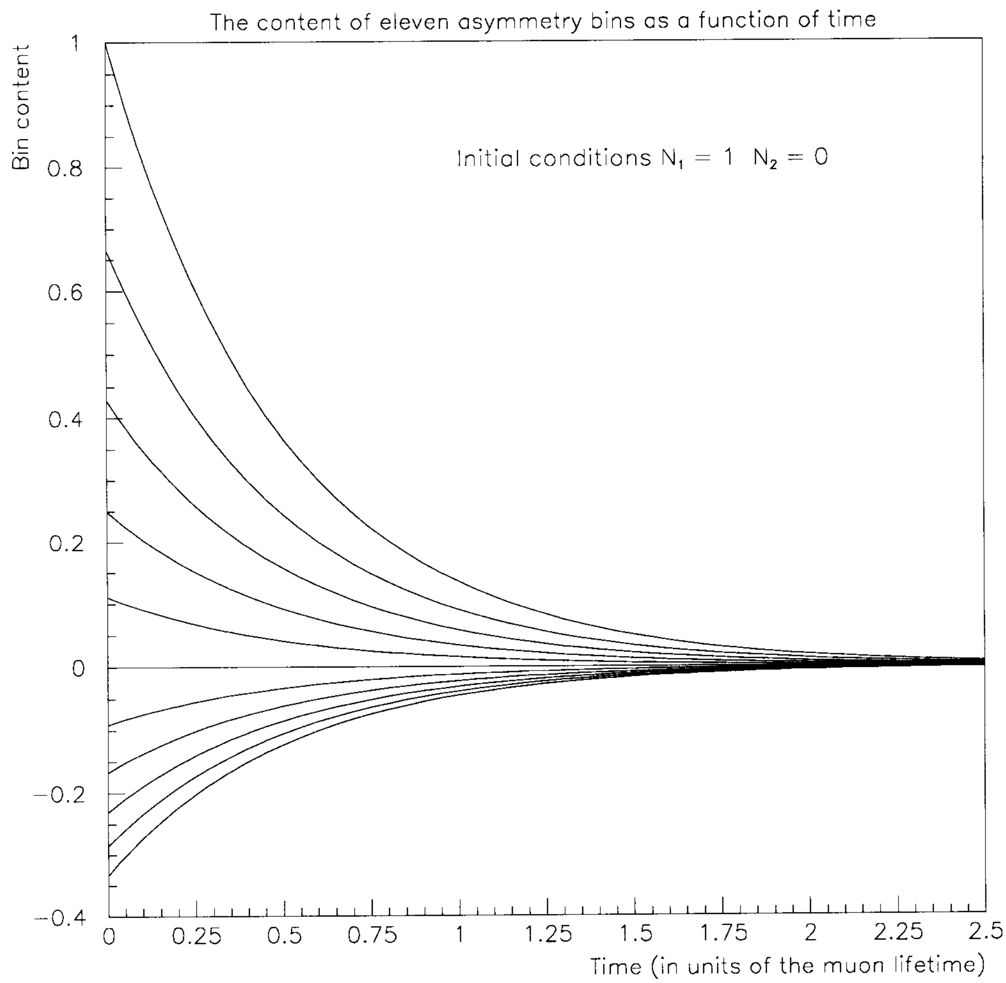

Figure 8: Content of eleven asymmetry bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

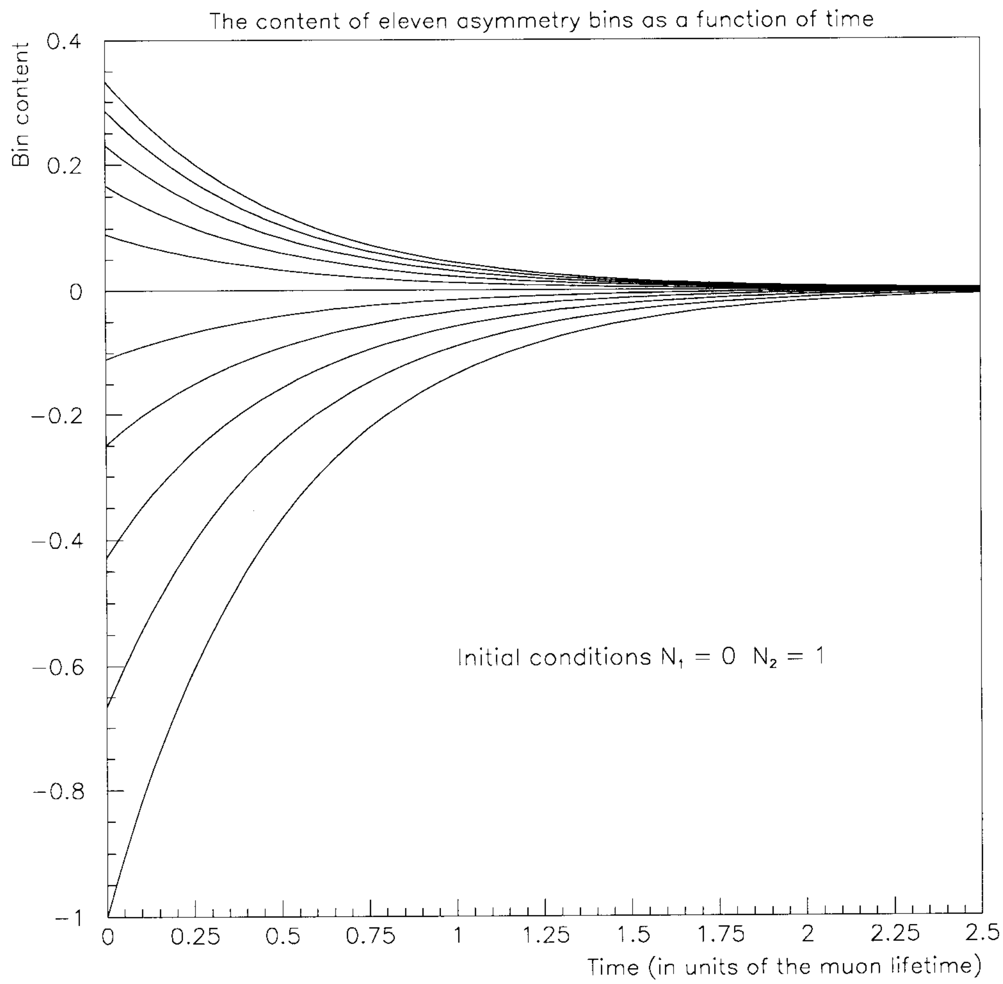

Figure 9: Content of eleven asymmetry bins. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

The following figures show the positron spectrum and positron asymmetry as functions of time for different initial conditions:

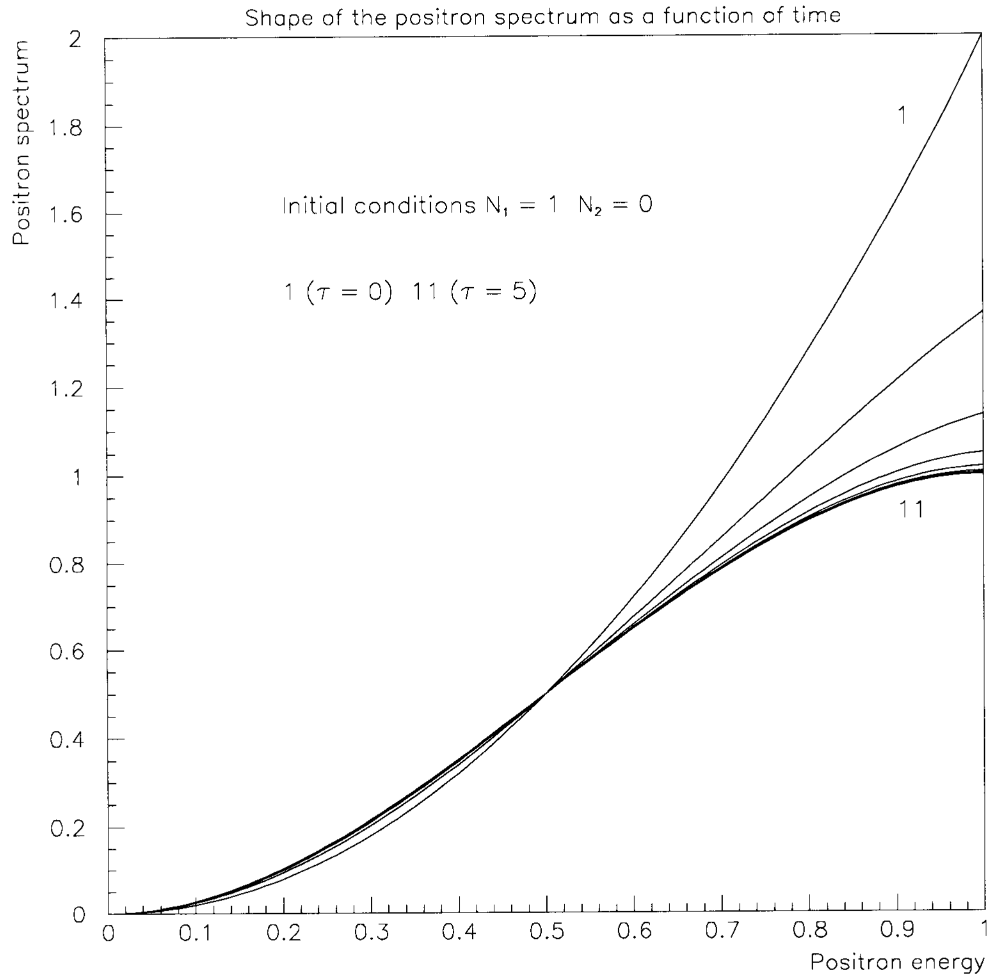

Figure 10: Positron spectrum. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

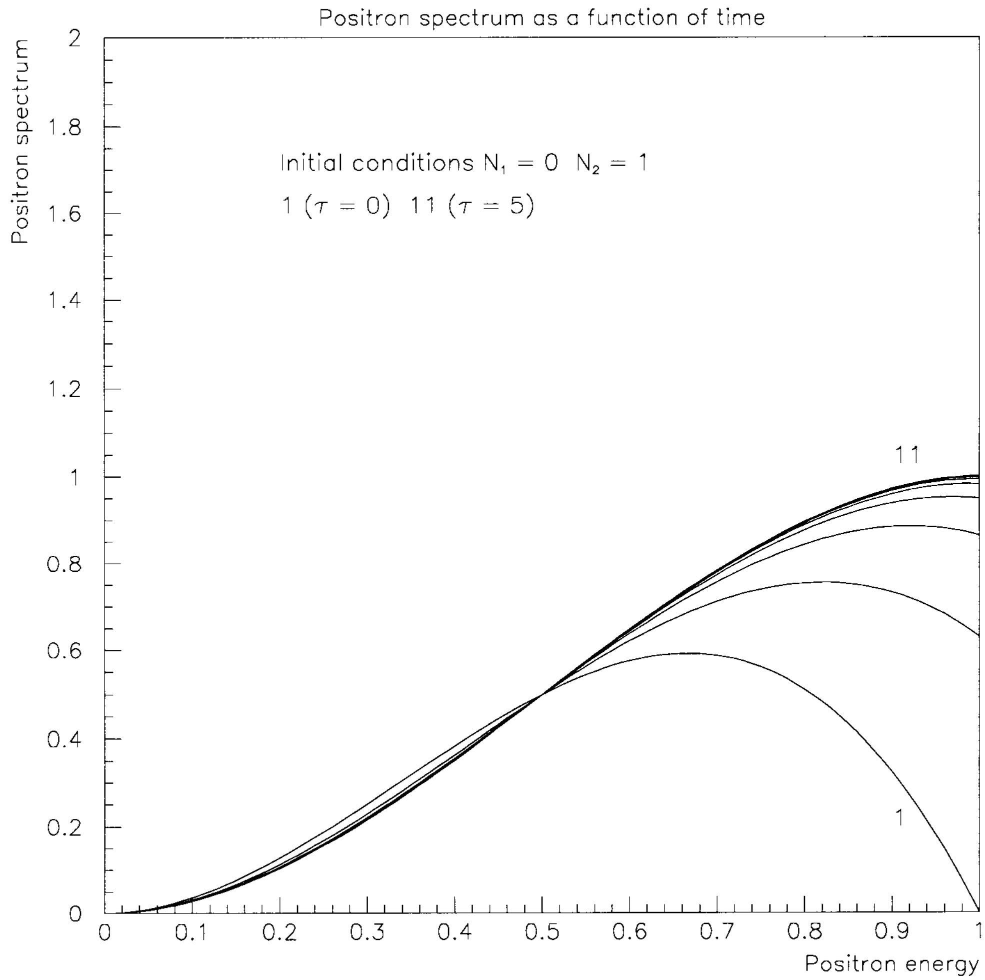

Figure 11: Positron spectrum. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

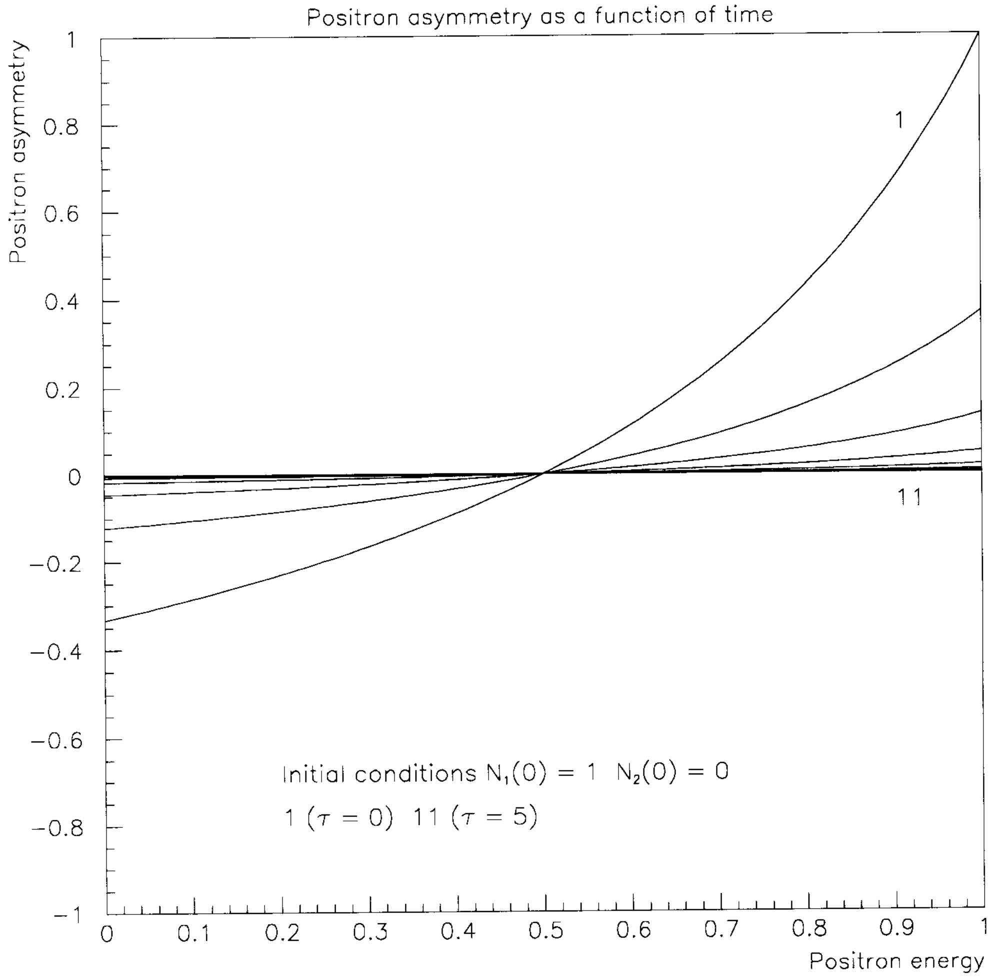

Figure 12: Positron asymmetry. Case $\alpha = 1.0$. Initial conditions: $N_1 = 1$, $N_2 = 0$.

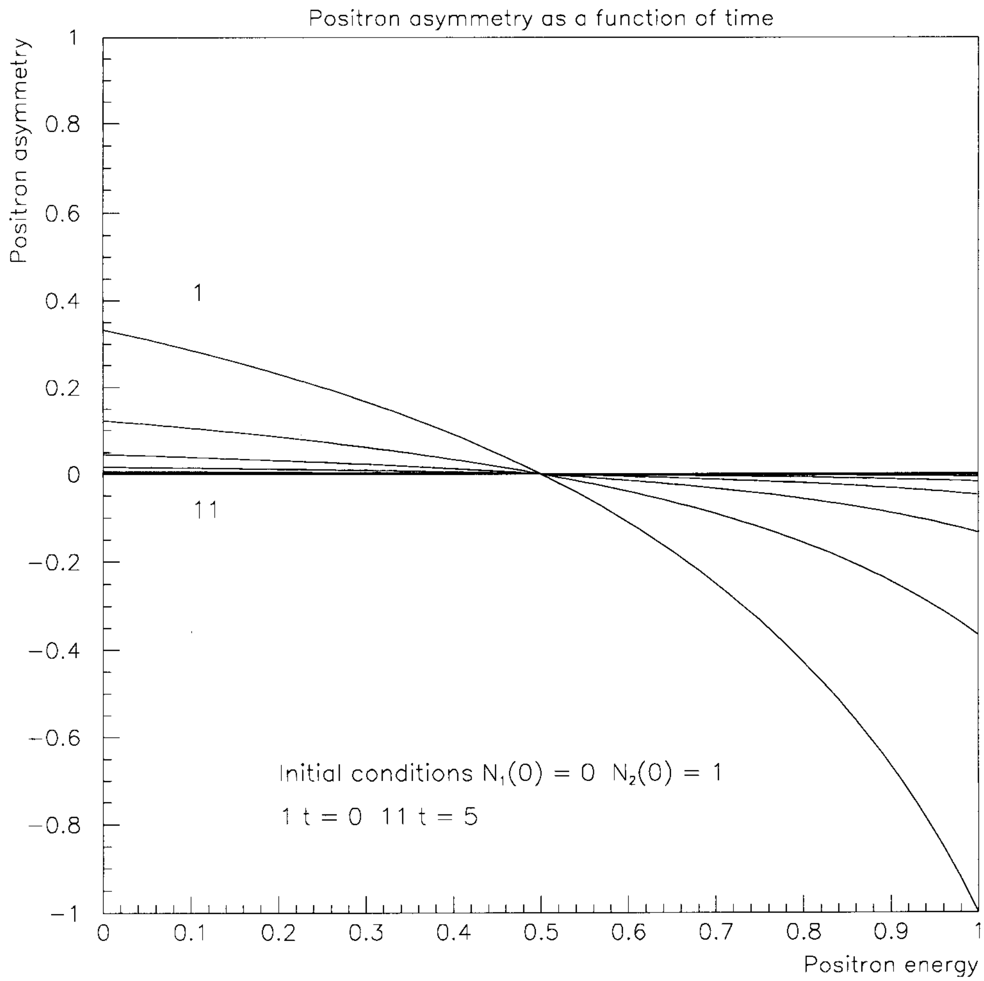

Figure 13: Positron asymmetry. Case $\alpha = 1.0$. Initial conditions: $N_1 = 0$, $N_2 = 1$.

November 29, 12:00