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Muon polarisation in electromagnetic fields

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1 Introduction

This note was originally written for the KEK-E246 experiment (Search for a transverse polarization of the muon in the $K^+ \to \pi^0\mu^+\nu$ decay), following extensive discussions with Peter Gumplinger who wrote the KEK-E246 and TRIUMF-E614 Monte Carlo codes, including the rotation of the muon polarization in a magnetic field by using the Bargmann-Michel-Telegdi (BMT) equation[1].

This note will also be of interest for the TRIUMF-E614 experiment. Its purpose is mainly pedagogical.

It starts with the description of the polarization in the rest system of the particle. It is based on the density matrix formalism and the Stokes vector (a three-vector).

Then the relativistic treatment of the polarization is presented. If the particle, initially at rest, is submitted to a Lorentz boost, the polarization vector becomes a four-vector. It reduces to a three-vector only in the rest system of the particle.

The Bargmann-Michel-Telegdi (BMT) equation, which describes the motion of the polarization vector (a four-vector) in an electromagnetic field, is introduced. An expression is obtained for the motion of the Stokes vector[2] (defined in the rest system of the particle, a three-vector).

As an application, the special case of a uniform magnetic field is considered and an analytical solution is given. A computer program based on this analytical solution has been written. This program has been used to test a subroutine of the Monte Carlo simulation, which applies to the more general case of a non-uniform (slowly varying) magnetic field.

In the experiments we are interested in, polarized muons are produced, then they go through several materials and through electromagnetic fields. Finally, they are slowed down to rest before they decay. One is therefore faced with several problems:

1. Knowledge of the initial polarization.
2. Rotation of the polarization vector in electromagnetic fields.
3. Depolarization, or change of magnitude of the polarization vector.

This note deals with the second problem, considering the case of a pure and uniform magnetic field. As said before its purpose is mainly pedagogical. It has also been used to check the Monte Carlo codes written by Peter Gumplinger for the KEK-E246 and TRIUMF-E614 experiments.

2 General formalism.

Let us consider an object with two polarization degrees of freedom. It can be a spin-1/2 particle (electron, muon, proton, etc...) but also a photon (even if in that case the situation is completely different). The general formalism for the description of the polarization of such an object has been discussed by Tolhoek [3].

The state of polarization can be written as:

$$|\phi\rangle = a_1|1\rangle + \ a_2|2\rangle \qquad \text{or} \qquad \phi = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \tag{1}$$

where a_1 and a_2 are complex numbers subject to the normalization condition:

$$|a_1|^2 + |a_2|^2 = 1 \tag{2}$$

$|1\rangle$ and $|2\rangle$ are two basic polarization states. For a spin-1/2 particle in its rest frame they could be spin-up and spin-down. For a photon they could be two circularly-polarized states of different helicities, or two linear polarization states orthogonal to each other.

One defines the expectation value $\vec{\xi} = \phi^\dagger \vec{\sigma} \phi$ (the components of $\vec{\sigma}$ are the Pauli matrices):

$$\xi_x = a_1 a_2^* + a_2 a_1^* \qquad \xi_y = i(a_1 a_2^* - a_2 a_1^*) \qquad \xi_z = a_1 a_1^* - a_2 a_2^* \tag{3}$$

These three real numbers are the components of a vector. It is well known that the spin-1/2 can be described either with a two-component spinor (SU_2) or a three-component vector (R_3). This vector has unit length:

$$\xi = \sqrt{\xi_x^2 + \xi_y^2 + \xi_z^2} = 1 \tag{4}$$

Hence two parameters are needed to describe the polarization of a spin-1/2 particle or a photon. Note that one uses the Pauli matrices for the photon, although it is not a spin-1/2 particle. The use of the Pauli (2×2) matrices is related to the fact that there are two polarisation states.

For a single particle this vector $\vec{\xi}$ carries information in terms of expectation values. Things become closer to physical reality if one considers ensembles of particles, like polarized beams or polarized targets.

For a beam of particles, or a target, one must take averages over an ensemble of particles. One introduces the density matrix:

$$\rho = \begin{pmatrix} \overline{a_1 a_1^*} & \overline{a_1 a_2^*} \\ \overline{a_2 a_1^*} & \overline{a_2 a_2^*} \end{pmatrix} = \frac{1}{2}\Big(1 + \vec{\sigma} \cdot \vec{\xi}\Big) \tag{5}$$

where $\vec{\xi}$ is a three-component (real) vector:

$$\xi_x = \overline{a_1 a_2^*} + \overline{a_2 a_1^*} \qquad \xi_y = i(\overline{a_1 a_2^*} - \overline{a_2 a_1^*}) \qquad \xi_z = \overline{a_1 a_1^*} - \overline{a_2 a_2^*} \tag{6}$$

The vector $\vec{\xi}$ now carries statistical information. It can be identified with the Stokes vector which was introduced many years ago (well before quantum mechanics and the discovery of the electron) to describe the polarization of light[2]. The three components of this vector are measurable quantities.

The Stokes vector does not have a unit length. Rather:

$$\xi^2 = 1 - 4(\overline{a_1 a_1^*}\ \overline{a_2 a_2^*} - \overline{a_1 a_2^*}\ \overline{a_2 a_1^*}) \tag{7}$$

According to the well-known inequality:

$$|\vec{a}_1 \cdot \vec{a}_2| \leq |\vec{a}_1|\ |\vec{a}_2| \tag{8}$$

one finds that:

$$\xi \leq 1 \tag{9}$$

For an unpolarized state, where the phases and the moduli in the ensemble are taken at random (subject to the normalization condition):

$$\overline{a_1 a_2^*} = \overline{a_2 a_1^*} = 0 \qquad \overline{a_1 a_1^*} = \overline{a_2 a_2^*} = \frac{1}{2} \tag{10}$$

And $\vec{\xi}$ is a null vector.

One can also write:

$$\vec{\xi} = \vec{n} P \tag{11}$$

where $\vec{n}$ is a unit vector which defines the state of polarization and P is the degree of polarization. Now, three parameters are needed to describe the polarization of an ensemble of spin-1/2 particles or photons.

For a pure state (all particles are in the same state of polarization) we can remove the horizontal bars:

$$P = 1 \qquad\qquad \rho = \frac{1}{2}\Big(1 + \vec{\sigma} \cdot \vec{n}\Big) \tag{12}$$

For an unpolarized state:

$$P = 0 \qquad\qquad \rho = \frac{1}{2} \tag{13}$$

For $0 < P < 1$ one has a state of partial polarization. Such a state can be obtained in several ways, for example:

1. by mixing incoherently a completely polarized ensemble and an unpolarized ensemble (with weights P and $1 - P$):

$$\rho = \frac{1}{2}\Big(1 + P\vec{\sigma}\cdot\vec{n}\Big) = P\frac{1}{2}\Big(1 + \vec{\sigma}\cdot n\Big) + \Big(1 - P\Big)\frac{1}{2} \tag{14}$$

2. by mixing incoherently two completely polarized ensembles of opposite polarizations (with weights $(1+P)/2$ and $(1-P)/2$):

$$\rho = \frac{1}{2}\Big(1 + P\vec{\sigma}\cdot\vec{n}\Big) = \frac{1+P}{2}\frac{1}{2}\Big(1 + \vec{\sigma}\cdot n\Big) + \frac{1-P}{2}\frac{1}{2}\Big(1 - \vec{\sigma}\cdot n\Big) \tag{15}$$

3 Motion of polarization in a magnetic field

The solution is well-known in the case of non-relativistic quantum mechanics. It results directly from the evolution equation:

$$i\frac{d}{dt}(\phi^\dagger\vec{\sigma}\phi) = \phi^\dagger[\vec{\sigma}, H]\phi \tag{16}$$

with:

$$H = \frac{eg}{2m}\vec{\sigma}\cdot\vec{\mathcal{B}} \tag{17}$$

where e is the elementary electric charge ($\pm|e|$) for positron and electron, respectively), g the gyromagnetic ratio, m the mass and $\vec{\mathcal{B}}$ the magnetic field. Hence:

$$\frac{d\vec{\xi}}{dt} = \frac{eg}{2m}(\vec{\mathcal{B}}\times\vec{\xi}) \tag{18}$$

where $\vec{\xi}$ is the Stokes vector. This is, in fact, a classical equation.

4 Relativistic case

Let us now consider the relativistic case. The precession of the polarization of particles moving in a homogeneous electromagnetic field has been studied by Bargmann, Michel and Telegdi [1]. The purpose of the paper was to give a fully relativistic treatment of the problem.

In the relativistic treatment one introduces two four-vectors (velocity and polarization):

$$u^\mu = \begin{pmatrix} u_0 \\ \vec{u} \end{pmatrix} \qquad\qquad u_\mu = \begin{pmatrix} u_0 \\ -\vec{u} \end{pmatrix} \qquad\qquad \text{(polar vector)} \tag{19}$$

with $u^0 = \gamma$ and $\vec{u} = \gamma\vec{v}$. And γ is the usual relativistic factor $\gamma = E/m$.

$$s^\mu = \begin{pmatrix} s_0 \\ \vec{s} \end{pmatrix} \qquad s_\mu = \begin{pmatrix} s_0 \\ -\vec{s} \end{pmatrix} \qquad \text{(axial vector)} \tag{20}$$

These four-vectors have unit length:

$$u^\mu u_\mu = 1 \qquad s^\mu s_\mu = -1 \tag{21}$$

and the following relation holds:

$$s^\mu u_\mu = 0 \tag{22}$$

or:

$$s^0 = \vec{s} \cdot \vec{v} \tag{23}$$

In the rest frame of the particle $\vec{v} = 0$ and $s^0 = 0$.

One has the following relations between the four-vector s^μ and the three-vector $\vec{\xi}$:

$$s^0 = \frac{\vec{\xi} \cdot \vec{p}}{m} \qquad \vec{s} = \vec{\xi} + \frac{(\vec{\xi} \cdot \vec{p})\ \vec{p}}{m\ (E+m)} \tag{24}$$

where $\vec{p}$ is the momentum of the particle and E its total energy. The latter equation can also be written as:

$$s^0 = \beta\gamma|\vec{\xi}_L| \qquad \vec{s} = \gamma\vec{\xi}_L + \vec{\xi}_T \tag{25}$$

where $\vec{\xi}_L$ and $\vec{\xi}_T$ are the longitudinal and transverse components of $\vec{\xi}$, along the momentum $\vec{p}$ and orthogonal to it, respectively. One recognizes the transformation property of a vector (pseudovector) under a Lorentz transformation.

One now has to obtain the relativistic equations. The relativistic equation of motion is:

$$\frac{du^\mu}{d\tau} = \frac{1}{m} f^\mu \tag{26}$$

where f is the four-vector force and τ is the proper time:

$$\tau = t/\gamma \tag{27}$$

For a particle in an electromagnetic field one has:

$$\frac{du^\mu}{d\tau} = \frac{e}{m}(F^{\mu\nu} u_\nu) \tag{28}$$

where $F^{\mu\nu}$ is the electromagnetic tensor. One could naively write for s^μ, by analogy with Equation 18:

$$\frac{ds^\mu}{d\tau} = \frac{eg}{2m}(F^{\mu\nu} s_\nu) \tag{29}$$

But it is easy to show that the equation:

$$\frac{ds^\mu}{d\tau}u_\mu + s^\mu\frac{du_\mu}{d\tau} = 0 \tag{30}$$

would not be satisfied. More details can be found in Jackson's book [4] (see also refs. [5, 6]). A recent paper [7] deals with the problem and gives interesting historical references.

One finally gets the BMT (Bargmann-Michel-Telegdi) equation[1]:

$$\frac{ds^\mu}{d\tau} = \frac{e}{m}\left[\frac{g}{2}F^{\mu\nu}s_\nu + \left(\frac{g}{2}-1\right)(s_\alpha F^{\alpha\beta}u_\beta)u^\mu\right] \tag{31}$$

The problem of spin rotation in an electromagnetic field has also been discussed by Lyuboshits[8]. He obtains a formula for the motion of the Stokes vector $\vec{\xi}$:

$$\frac{d\xi}{dt} = \vec{\Omega}\times\vec{\xi} \tag{32}$$

where:

$$\vec{\Omega}(t) = -\frac{e}{2m_0c}g\left[\vec{\mathcal{B}} - \frac{\gamma-1}{\gamma}\left(\frac{\vec{v}}{v}\right)\left(\vec{\mathcal{B}}\cdot\frac{\vec{v}}{v}\right) + \left(\vec{\mathcal{E}}\times\frac{\vec{v}}{c}\right)\right] - (\gamma-1)\left[\left(\frac{\vec{v}}{v}\right)\times\frac{d}{dt}\left(\frac{\vec{v}}{v}\right)\right] \tag{33}$$

Or equivalently:

$$\begin{aligned}\vec{\Omega}(t) &= -\frac{eg}{2m_0c\gamma}\left[\vec{\mathcal{B}}_L + \gamma\vec{\mathcal{B}}_T + \gamma\left(\vec{\mathcal{E}}\times\frac{\vec{v}}{c}\right)\right] - (\gamma-1)\left[\left(\frac{\vec{v}}{v}\right)\times\frac{d}{dt}\left(\frac{\vec{v}}{v}\right)\right]\\ &= -\frac{eg}{2m_0c\gamma}\vec{\mathcal{B}}^* - (\gamma-1)\left[\left(\frac{\vec{v}}{v}\right)\times\frac{d}{dt}\left(\frac{\vec{v}}{v}\right)\right]\end{aligned} \tag{35}$$

Where $\vec{\mathcal{B}}_L$ and $\vec{\mathcal{B}}_T$ are the longitudinal and transverse components of the magnetic field (with respect to the particle velocity) and $\vec{\mathcal{B}}^*$ the magnetic field in the rest system of the particle. Therefore the interpretation of the first term is simple. The second term represents the Thomas precession[9].

It is not difficult to show that this formula is equivalent to the one given by Buon[10]:

$$\vec{\Omega}(t) = -\frac{e}{m_0c\gamma}\left[(1+a\gamma)\vec{\mathcal{B}}_T + (1+a)\vec{\mathcal{B}}_L + \left(a+\frac{1}{\gamma+1}\right)\left(\vec{\mathcal{E}}\times\frac{\vec{v}}{c}\right)\gamma\right] \tag{36}$$

where a is the anomaly:[11]

$$a = \frac{g}{2} - 1 = 0.0011659160 \pm 0.0000000006 \tag{37}$$

5 Special case: pure magnetic field

5.1 Rotation of the velocity

One uses $c = 1$.

$$\frac{d\vec{p}}{dt} = e\left(\vec{v}\times\vec{\mathcal{B}} + \vec{\mathcal{E}}\right) \qquad\qquad \frac{dE}{dt} = e\vec{v}\cdot\vec{\mathcal{E}} \tag{38}$$

Here e is the elementary charge. It can be positive or negative.

If there is no electric field:

$$\frac{d\vec{p}}{dt} = e\left(\vec{v} \times \vec{\mathcal{B}}\right) \qquad (\vec{p} = m_0\gamma\vec{v}) \qquad \frac{dE}{dt} = 0 \tag{39}$$

$$\vec{p} \cdot \frac{d\vec{p}}{dt} = em_0\gamma\left[\vec{v} \cdot (\vec{v} \times \vec{\mathcal{B}})\right] = 0 \tag{40}$$

Therefore $|\vec{p}|$, E, $|\vec{v}|$ and γ are constant in time. And:

$$\frac{d\vec{v}}{dt} = \frac{e}{m_0\gamma}(\vec{v} \times \vec{\mathcal{B}}) \tag{41}$$

In a homogeneous (and independent of time) magnetic field $\vec{\mathcal{B}} = \vec{N}\mathcal{B}$, where $\vec{N}$ is a unit vector:

$$\frac{d\vec{v}}{dt} = \frac{e\mathcal{B}}{m_0\gamma}(\vec{v} \times \vec{N}) = i\omega(\vec{\Sigma} \cdot \vec{N})\vec{v} \qquad (\omega = \frac{e\mathcal{B}}{m_0\gamma}) \tag{42}$$

Note that ω_0 and ω can be positive or negative.

The vector $\vec{\Sigma}$ is defined as:

$$\Sigma_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix} \qquad \Sigma_y = \begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad \Sigma_z = \begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{43}$$

The solution is:

$$\vec{v}(t) = \exp\left[i\omega(\vec{\Sigma} \cdot \vec{N})t\right]\vec{v}_0 = \left[1 + i(\vec{\Sigma} \cdot \vec{N})\sin\omega t - (\vec{\Sigma} \cdot \vec{N})^2(1 - \cos\omega t)\right]\vec{v}_0 \tag{44}$$

Let us introduce the unit vectors:

$$\vec{k}(t) = \frac{\vec{v}}{v_0} \qquad \vec{k}_0 = \frac{\vec{v}_0}{v_0} \tag{45}$$

Then:

$$\vec{k}(t) = \left[1 + i(\vec{\Sigma} \cdot \vec{N})\sin\omega t - (\vec{\Sigma} \cdot \vec{N})^2(1 - \cos\omega t)\right]\vec{k}_0 \tag{46}$$

Let us define the angle χ ($0 \leq \chi \leq \pi$):

$$\vec{N} \cdot \vec{k} = \vec{N} \cdot \vec{k}_0 = \cos\chi \tag{47}$$

Then:

$$\mathcal{B}_L = \mathcal{B}\cos\chi \qquad \mathcal{B}_T = \mathcal{B}\sin\chi \tag{48}$$

One gets:

$$\vec{k}(t) = \vec{k}_0\cos\omega t + \vec{N}\cos\chi(1 - \cos\omega t) - (\vec{N} \times \vec{k}_0)\sin\omega t \tag{49}$$

Let us define two unit vectors:

$$\frac{\vec{k}_0 - \vec{N}\cos\chi}{\sin\chi} = \vec{P} \tag{50}$$

$$\frac{\vec{N}\times\vec{k}_0}{\sin\chi} = \vec{Q} \tag{51}$$

It is easy to verify that $\vec{N}$, $\vec{P}$ and $\vec{Q}$ form an orthogonal system of reference. Then:

$$\vec{k}(t) = \vec{N}\cos\chi + (\vec{P}\cos\omega t - \vec{Q}\sin\omega t)\sin\chi \tag{52}$$

The integration gives:

$$\vec{r}(t) = \left(\frac{v_0}{\omega}\right)\left\{\vec{N}\omega t\cos\chi + \left[\vec{P}\sin\omega t - \vec{Q}(1-\cos\omega t)\sin\chi\right]\right\} \tag{53}$$

given that for $t = 0$ one has $\vec{r} = 0$.

The angle χ is constant and also the magnitude of $\vec{k}$:

$$|\vec{k}| = k = k_0 \tag{54}$$

The vector $\vec{k}$ precesses around the vector $\vec{N}$ at the frequency $\omega/2\pi$.

5.2 Rotation of the polarization vector

$$\frac{d\vec{\xi}}{dt} = \vec{\xi}\times\vec{\Omega} \tag{55}$$

with[10]:

$$\vec{\Omega}(t) = \frac{e}{m_0\gamma}\left[(1+a)\vec{\mathcal{B}}_L + (1+\gamma a)\vec{\mathcal{B}}_T\right] = \frac{e}{m_0\gamma}\left[(1+\gamma a)\vec{\mathcal{B}} + \frac{a(1-\gamma)}{v^2}(\vec{\mathcal{B}}\cdot\vec{v})\vec{v}\right] \tag{56}$$

$$= \omega\left[(1+\gamma a)\vec{N} - a(\gamma-1)\vec{k}\cos\chi\right] \tag{57}$$

Here again the electric charge e and the frequency ω can be positive or negative.

We recall the expression of $\vec{k}$:

$$\vec{k}(t) = \vec{N}\cos\chi + (\vec{P}\cos\omega t - \vec{Q}\sin\omega t)\sin\chi \tag{58}$$

The modulus $\Omega = |\vec{\Omega}|$ is constant in time:

$$\Omega^2 = \omega^2\left[(1+\gamma a)^2 - a(\gamma-1)(2+a+\gamma a)\cos^2\chi\right]$$

$$= \omega^2\left[(1+a)^2 + a(\gamma-1)(2+a+\gamma a)\sin^2\chi\right] \tag{60}$$

The expression for $\vec{\Omega}$ can be written:

$$\vec{\Omega}(t) = \omega\left\{\vec{N} + a\left[(\cos^2\chi + \gamma\sin^2\chi)\vec{N} - (\gamma-1)\sin\chi\cos\chi(\vec{P}\cos\omega t - \vec{Q}\sin\omega t)\right]\right\} \tag{61}$$

We define:

$$\lambda = \omega a\sqrt{\cos^2\chi + \gamma^2\sin^2\chi} \tag{62}$$

$$\cos\eta = \frac{\cos^2\chi + \gamma\sin^2\chi}{\sqrt{\cos^2\chi + \gamma^2\sin^2\chi}} \qquad \sin\eta = \frac{(\gamma-1)\sin\chi\cos\chi}{\sqrt{\cos^2\chi + \gamma^2\sin^2\chi}} \tag{63}$$

This gives:

$$\vec{\Omega} = \omega\vec{N} + \lambda\left[\vec{N}\cos\eta - (\vec{P}\cos\omega t - \vec{Q}\sin\omega t)\sin\eta\right] \tag{64}$$

The length of the vector $\vec{\Omega}$ is given by:

$$\Omega^2 = \omega^2 + 2\lambda\omega\cos\eta + \lambda^2 \tag{65}$$

It is independent of time.

One can write:

$$\frac{d\vec{\Omega}}{dt} = \lambda\omega(\vec{P}\sin\omega t + \vec{Q}\cos\omega t)\sin\eta \tag{66}$$

and:

$$\vec{N}\times\vec{\Omega} = -\lambda(\vec{P}\sin\omega t + \vec{Q}\sin\omega t)\sin\eta \tag{67}$$

Therefore:

$$\frac{d\vec{\Omega}}{dt} = \omega(\vec{\Omega}\times\vec{N}) \tag{68}$$

The vector $\vec{\Omega}$ precesses around the vector $\vec{N}$. The angle κ between $\vec{\Omega}$ and $\vec{N}$ is given by:

$$\cos\kappa = \frac{\omega + \lambda\cos\eta}{\sqrt{\omega^2 + 2\lambda\omega\cos\eta + \lambda^2}} \tag{69}$$

and in the reference system formed by the vectors $\vec{N}$, $\vec{P}$ and $\vec{Q}$, the components of $\vec{\Omega} - \omega\vec{N}$ are:

$$\vec{\Omega} - \omega\vec{N} = \lambda\vec{M} \tag{70}$$

where $\vec{M}$ is a unit vector:

$$\vec{M} = \begin{pmatrix} \cos\eta \\ -\sin\eta\cos\omega t \\ \sin\eta\sin\omega t \end{pmatrix} \tag{71}$$

It can be shown that the vector $\vec{\xi}$ makes a constant angle with the vector $\vec{M}$:

$$\frac{d}{dt}\left[(\vec{\Omega}-\omega\vec{N})\cdot\vec{\xi}\right]=\frac{d\vec{\Omega}}{dt}\cdot\vec{\xi}+(\vec{\Omega}-\omega\vec{N})\cdot\frac{d\vec{\xi}}{dt} \tag{72}$$

$$=\omega(\vec{\Omega}\times\vec{N})\cdot\vec{\xi}+(\vec{\Omega}-\vec{N})\cdot(\vec{\xi}\times\Omega)=\omega\left[(\vec{\Omega}\times\vec{N})\cdot\vec{\xi}-\vec{N}\cdot(\vec{\xi}\times\vec{\Omega})\right]=0 \tag{73}$$

Therefore the motion of the polarization vector can be depicted in the following way:

- The vector $\vec{\Omega}$ precesses around the magnetic field at the frequency $\omega/2\pi$.
- The polarization vector $\vec{\xi}$ precesses around the vector $\vec{M}$ at the frequency $\lambda/2\pi$.

5.3 An analytical solution

The motion of the polarization vector is given by:

$$\frac{d\vec{\xi}}{dt}=\vec{\xi}\times\vec{\Omega}=i(\vec{\Sigma}\cdot\vec{\Omega})\vec{\xi} \tag{74}$$

(the $\vec{\Sigma}$ matrices have been defined before).
With:

$$\vec{\Omega}=\omega\vec{N}+\lambda\vec{M} \tag{75}$$

the equation of motion contains two terms:

$$\frac{d\vec{\xi}}{dt}=\vec{\xi}\times\vec{\Omega}=i(\vec{\Sigma}\cdot\vec{\Omega})\vec{\xi}=i\omega(\vec{\Sigma}\cdot\vec{N})+i\lambda(\vec{\Sigma}\cdot\vec{M}) \tag{76}$$

In the reference system formed by the vectors $\vec{N}$, $\vec{P}$ and $\vec{Q}$, the components of $\vec{\Omega}$ are:

$$\vec{\Omega}(t)=\begin{pmatrix}\omega+\lambda\cos\eta\\ -\lambda\sin\eta\cos\omega t\\ \lambda\sin\eta\sin\omega t\end{pmatrix} \tag{77}$$

Let us define a new vector $\vec{\xi'}$ such that:

$$\vec{\xi'}=A\vec{\xi}\qquad\qquad \vec{\xi}=A^{-1}\vec{\xi'} \tag{78}$$

The equation of motion for $\vec{\xi'}$ writes:

$$\frac{d\vec{\xi'}}{dt}=\left[iA(\vec{\Sigma}\cdot\vec{\Omega})A^{-1}-A\frac{dA^{-1}}{dt}\right]\vec{\xi'} \tag{79}$$

Let us take:

$$A(t)=\begin{pmatrix}1&0&0\\0&\cos\omega t&-\sin\omega t\\0&\sin\omega t&\cos\omega t\end{pmatrix} \tag{80}$$

After some calculations:

$$\frac{d\vec{\xi}'}{dt} = \lambda \begin{pmatrix} 0 & 0 & \sin\eta \\ 0 & 0 & \cos\eta \\ -\sin\eta & -\cos\eta & 0 \end{pmatrix} \vec{\xi}' = i\lambda(\vec{\Sigma}\cdot\vec{R})\vec{\xi}' \tag{81}$$

with:

$$\vec{R} = \begin{pmatrix} \cos\eta \\ -\sin\eta \\ 0 \end{pmatrix} \tag{82}$$

The solution is:

$$\vec{\xi}'(t) = \exp\Big[i\lambda(\vec{\Sigma}\cdot\vec{R})\Big]\vec{\xi}'(0) = \Big[1 + i(\vec{\Sigma}\cdot\vec{R})\sin\lambda t + (\vec{\Sigma}\cdot\vec{R})^2(\cos\lambda t - 1)\Big]\vec{\xi}'(0) \tag{83}$$

Since $A(0) = 1$, $\vec{\xi}'(0) = \vec{\xi}(0)$. Therefore:

$$\vec{\xi}(t) = \Big[A^{-1} + iA^{-1}(\vec{\Sigma}\cdot\vec{R})\sin\lambda t + A^{-1}(\vec{\Sigma}\cdot\vec{R})^2(\cos\lambda t - 1)\Big]\vec{\xi}(0) \tag{84}$$

and:

$$\begin{aligned}\vec{\xi}(t) = \Bigg[& \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega t & \sin\omega t \\ 0 & -\sin\omega t & \cos\omega t \end{pmatrix} \\ & + \sin\lambda t \begin{pmatrix} 0 & 0 & \sin\eta \\ -\sin\eta\sin\omega t & -\cos\eta\sin\omega t & \cos\eta\cos\omega t \\ -\sin\eta\cos\omega t & -\cos\eta\cos\omega t & -\cos\eta\sin\omega t \end{pmatrix} \\ & +(\cos\lambda t - 1) \begin{pmatrix} \sin^2\eta & \sin\eta\cos\eta & 0 \\ \sin\eta\cos\eta\cos\omega t & \cos^2\eta\cos\omega t & \sin\omega t \\ -\sin\eta\cos\eta\sin\omega t & -\cos^2\eta\sin\omega t & \cos\omega t \end{pmatrix} \Bigg]\vec{\xi}(0)\end{aligned} \tag{87}$$

Finally, we get for the components of $\vec{\xi}(t)$:

$$\begin{aligned}\xi_n(t) &= \xi_n(0)\Big[1 + \sin^2\eta(\cos\lambda t - 1)\Big] \\ &+\xi_p(0)\Big[\sin\eta\cos\eta(\cos\lambda t - 1)\Big] \\ &+\xi_q(0)\Big[\sin\eta\sin\lambda t\Big]\end{aligned} \tag{88}$$

$$\begin{aligned}\xi_p(t) &= \xi_n(0)\Big[-\sin\eta\sin\omega t\sin\lambda t + \sin\eta\cos\eta\cos\omega t(\cos\lambda t - 1)\Big] \\ &+\xi_p(0)\Big[\cos\omega t - \cos\eta\sin\omega t\sin\lambda t + \cos^2\eta\cos\omega t(\cos\lambda t - 1)\Big] \\ &+\xi_q(0)\Big[\sin\omega t + \cos\eta\cos\omega t\sin\lambda t + \sin\omega t(\cos\lambda t - 1)\Big]\end{aligned} \tag{89}$$

$$
\begin{aligned}
\xi_q(t) &= \xi_n(0)\Big[-\sin\eta\cos\omega t\sin\lambda t - \sin\eta\cos\eta\sin\omega t(\cos\lambda t - 1)\Big] \\
&\quad +\xi_p(0)\Big[-\sin\omega t - \cos\eta\cos\omega t\sin\lambda t - \cos^2\eta\sin\omega t(\cos\lambda t - 1)\Big] \\
&\quad +\xi_q(0)\Big[\cos\omega t - \cos\eta\sin\omega t\sin\lambda t + \cos\omega t(\cos\lambda t - 1)\Big]
\end{aligned}
\tag{90}
$$

To get the polarization components in the (x, y, z) frame one has to use the transformation:

$$
\begin{pmatrix} \xi_x(t) \\ \xi_y(t) \\ \xi_z(t) \end{pmatrix} = \begin{pmatrix} N_x & P_x & Q_x \\ N_y & P_y & Q_y \\ N_z & P_z & Q_z \end{pmatrix} \begin{pmatrix} \xi_n(t) \\ \xi_p(t) \\ \xi_q(t) \end{pmatrix}
\tag{91}
$$

The initial polarization can be transformed from the (x, y, z) frame to the (n, p, q) frame by using the inverse matrix.

We now calculate the longitudinal polarization. The vector $\vec{k}$ can be written:

$$
\vec{k} = k_n\vec{N} + k_p\vec{P} + k_q\vec{Q}
\tag{92}
$$

with:

$$
\begin{aligned}
k_n &= \cos\chi \\
k_p &= +\sin\chi\cos\omega t \\
k_q &= -\sin\chi\sin\omega t
\end{aligned}
\tag{93}
$$

The longitudinal polarisation is:

$$
P_l(t) = \vec{k}\cdot\vec{\xi} = k_n\xi_n(t) + k_p\xi_p(t) + k_q\xi_q(t)
\tag{94}
$$

or:

$$
\begin{aligned}
P_l(t) &= \xi_n(0)\Big[\cos\chi\sin\eta\sin(\eta+\chi)\cos\lambda t\Big] \\
&\quad +\xi_p(0)\Big[\sin\chi + \cos\eta\sin(\eta+\chi)(\cos\lambda t - 1)\Big] \\
&\quad +\xi_q(0)\Big[\sin(\eta+\chi)\sin\lambda t\Big]
\end{aligned}
\tag{95}
$$

For $t = 0$:

$$
P_l(0) = \xi_n(0)\cos\chi + \xi_p(0)\sin\chi
\tag{96}
$$

In fact:

$$
\vec{\xi}(0) = \xi_p(0)\vec{P} + \xi_n(0)\vec{N}
\tag{97}
$$

$$
\vec{\xi}(0)\cdot\vec{k}_0 = \xi_n(0)\cos\chi + \xi_p(0)\sin\chi
\tag{98}
$$

and finally:

$$
P_l(t) = P_l(0) + \sin(\eta+\chi)\Big\{\Big[\xi_n(0)\sin\eta + \xi_p(0)\cos\eta\Big](\cos\lambda t - 1) + \xi_q(0)\sin\lambda t\Big\}
\tag{99}
$$

5.4 Special cases

5.4.1 No anomaly

We have $a = 0$ and therefore $\lambda = 0$. The formulae simplify to:

$$\begin{aligned}
\xi_n(t) &= \xi_n(0) \\
\xi_p(t) &= \xi_p(0)\cos\omega t + \xi_q(0)\sin\omega t \\
\xi_q(t) &= -\xi_p(0)\sin\omega t + \xi_q(0)\cos\omega t \\
P_l(t) &= P_l(0)
\end{aligned} \tag{100}$$

5.4.2 Initial velocity orthogonal to magnetic field

We have $\chi = \pi/2$, $\lambda = \omega\gamma a$, $\eta = 0$.

$$\begin{aligned}
\xi_n(t) &= \xi_n(0) \\
\xi_p(t) &= \xi_p(0)\cos(1+\gamma a)\omega t + \xi_q(0)\sin(1+\gamma a)\omega t \\
\xi_q(t) &= -\xi_p(0)\sin(1+\gamma a)\omega t + \xi_q(0)\cos(1+\gamma a)\omega t \\
P_l(t) &= P_l(0) + \left[\xi_p(0)(\cos\lambda t - 1) + \xi_q(0)\sin\lambda t\right]
\end{aligned} \tag{101}$$

5.4.3 Initial velocity along magnetic field

We have $\chi = 0$, $\lambda = \omega a$, $\eta = 0$.

$$\begin{aligned}
\xi_n(t) &= \xi_n(0) \\
\xi_p(t) &= \xi_p(0)\cos(1+a)\omega t + \xi_q(t)\sin(1+a)\omega t \\
\xi_q(t) &= -\xi_p(0)\sin(1+a)\omega t + \xi_q(0)\cos(1+a)\omega t \\
P_l(t) &= P_l(0)
\end{aligned} \tag{102}$$

5.4.4 Particle at rest

We have $\gamma = 1$ and therefore $\lambda = \omega a$. The polarization vector precesses around the magnetic field at the frequency $\omega(1+a)/2\pi$.

5.5 The program POLAR

This program calculate the rotation of the polarization vector in a constant (and time-independent) magnetic field.

The input data are:

- the number of values to be printed.
- the value of γ (relativistic factor), the value of the anomaly a and the value of the Larmor frequency ($\omega_0 = e\mathcal{B}/m_0$).
- the polar angles (α, β) of the initial velocity.

- the polar angles (θ, ϕ) of the magnetic field.
- the polar angles (ψ, ζ) of the initial polarization vector.

The conventions for (α, θ, ψ) and (β, ϕ, ζ) are as follows:

Axis	α, θ, ψ	β, ϕ, ζ
x	90^o	0^o
y	90^o	90^o
z	0^o	any value

The output data are functions of the angle ωt:

- the components of the velocity in the laboratory frame.
- the coordinates of the particle in the laboratory frame.
- the components of the polarization vector $\vec{\xi}$ in the rest frame of the particle.

5.6 Data

From "Review of Particle Physics":[11] electron cyclotron frequency/field.

$$\omega^e_{cycl}/\mathcal{B} = e/m_e = 1.758820174(71) \times 10^{11} \text{ rad s}^{-1} \text{ T}^{-1} \tag{103}$$

Using the electron and muon masses:

$$m_e = 0.510998902 \text{ MeV} \tag{104}$$

$$m_\mu = 105.658357 \text{ MeV} \tag{105}$$

One gets:

$$\omega^\mu_{cycl}/\mathcal{B} = e/m_\mu = 0.85062460 \times 10^9 \text{ rad s}^{-1} \text{ T}^{-1} \tag{106}$$

6 Appendix: the "magic" momentum

In $(g-2)$ experiments[12] a special value of the muon momentum is chosen in order to eliminate the effect of the electric field which is used in the accelerator for vertical focussing.

6.1 Rotation of the particle velocity

In the presence of transverse electric and magnetic fields (orthogonal to the particle velocity):

$$\vec{v} \cdot \vec{\mathcal{B}} = \vec{v} \cdot \vec{\mathcal{E}} = 0 \tag{107}$$

the equations describing the particle motion:

$$\frac{d\vec{p}}{dt} = e\left(\vec{v} \times \vec{\mathcal{B}} + \vec{\mathcal{E}}\right) \qquad\qquad \frac{dE}{dt} = e(\vec{v} \cdot \vec{\mathcal{E}}) \tag{108}$$

can be simplified (introducing $\vec{\beta} = \vec{v}$ and $E = m_0\gamma\vec{\beta}$, $c = 1$):

$$m_0\frac{d\gamma}{dt} = 0 \qquad \gamma = \text{ constant} \qquad m_0\gamma\frac{d\vec{\beta}}{dt} = e\left(\vec{\beta} \times \vec{\mathcal{B}} + \vec{\mathcal{E}}_T\right) \tag{109}$$

Using the identity:

$$\vec{\beta} \times (\vec{\beta} \times \vec{\mathcal{E}}) = \vec{\beta}\ (\vec{\beta} \cdot \vec{\mathcal{E}}) - \beta^2\ \vec{\mathcal{E}} \tag{110}$$

The longitudinal and transverse components of the electric field are:

$$\begin{aligned} \vec{\mathcal{E}}_L &= \frac{1}{\beta^2}\ \vec{\beta}\ (\vec{\beta} \cdot \vec{\mathcal{E}}) \\ \vec{\mathcal{E}}_T &= \vec{\mathcal{E}} - \frac{1}{\beta^2}\ \vec{\beta}\ (\vec{\beta} \cdot \vec{\mathcal{E}}) = -\frac{1}{\beta^2}\ \vec{\beta} \times (\vec{\beta} \times \vec{\mathcal{E}}) \end{aligned} \tag{111}$$

$$m_0\gamma\frac{d\vec{\beta}}{dt} = e\left(\vec{\beta} \times \vec{\mathcal{B}} - \frac{1}{\beta^2}\ \vec{\beta} \times (\vec{\beta} \times \vec{\mathcal{E}})\right) = e\ \vec{\beta} \times \left(\vec{\mathcal{B}} - \frac{1}{\beta^2}\ (\vec{\beta} \times \vec{\mathcal{E}})\right) \tag{112}$$

$$\frac{d\vec{\beta}}{dt} = \frac{e}{m_0\gamma}\ \vec{\beta} \times \left(\vec{\mathcal{B}} - \frac{1}{\beta^2}\ (\vec{\beta} \times \vec{\mathcal{E}})\right) \tag{113}$$

or:

$$\frac{d\vec{\beta}}{dt} = \vec{\beta} \times \vec{\omega}_c \tag{114}$$

with:

$$\vec{\omega}_c = \frac{e}{m_0\gamma}\left(\vec{\mathcal{B}} - \frac{1}{\beta^2}\ (\vec{\beta} \times \vec{\mathcal{E}})\right) \tag{115}$$

and, since:

$$\frac{1}{\beta^2} = \frac{\gamma^2}{\gamma^2 - 1} \tag{116}$$

$$\vec{\omega}_c = \frac{e}{m_0\gamma}\left(\vec{\mathcal{B}} - \frac{\gamma^2}{\gamma^2 - 1}\ (\vec{\beta} \times \vec{\mathcal{E}})\right) \tag{117}$$

To be compared with Farley and Picasso[13], page 266:

$$\vec{\omega}_c = \frac{e}{m_0}\left(\frac{\vec{\mathcal{B}}}{\gamma} - \frac{\gamma}{\gamma^2 - 1}\ (\vec{\beta} \times \vec{\mathcal{E}})\right) \tag{118}$$

6.2 Rotation of the polarization vector

The formula for the rotation of the polarization vector is[10], for a positive muon:

$$\vec{\omega}_s = \frac{e}{m_0\gamma}\left[(1+a\gamma)\vec{\mathcal{B}}_T + (1+a)\vec{\mathcal{B}}_L + \left(a + \frac{1}{\gamma+1}\right)\left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c}\right)\gamma\right] \tag{119}$$

and with $\vec{\mathcal{B}}_L = 0$, $\vec{\mathcal{B}}_T = \vec{\mathcal{B}}$, and changing notations:

$$\vec{\omega}_s = \frac{e}{m_0\gamma}\left[(1+a\gamma)\vec{\mathcal{B}} - \left(a + \frac{1}{\gamma+1}\right)\left(\vec{\beta} \times \vec{\mathcal{E}}\right)\gamma\right] \tag{120}$$

where a is the anomaly:[11]

$$a = \frac{g}{2} - 1 = 0.0011659160 \pm 0.0000000006 \tag{121}$$

to be compared with Farley and Picasso[13], page 266:

$$\vec{\omega}_s = \frac{e}{m_0}\left[\frac{\vec{\mathcal{B}}}{\gamma} - \frac{1}{\gamma+1}(\vec{\beta} \times \vec{\mathcal{E}}) + a\left(\vec{\mathcal{B}} - (\vec{\beta} \times \vec{\mathcal{E}})\right)\right] \tag{122}$$

6.3 The "magic momentum"

The difference between the two frequencies is:

$$\Delta\omega = \omega_s - \omega_c = \frac{e}{m_0 c\gamma}\left[(1+a\gamma)\vec{\mathcal{B}} - \left(a + \frac{1}{\gamma+1}\right)\left(\vec{\beta} \times \vec{\mathcal{E}}\right)\gamma - \vec{\mathcal{B}} + \frac{\gamma^2}{\gamma^2-1}(\vec{\beta} \times \vec{\mathcal{E}})\right] \tag{123}$$

$$\Delta\omega = \frac{e}{m_0}\left[a\vec{\mathcal{B}} - \left(a + \frac{1}{\gamma+1} - \frac{\gamma}{\gamma^2-1}\right)\left(\vec{\beta} \times \vec{\mathcal{E}}\right)\right] \tag{124}$$

$$\Delta\omega = \frac{e}{m_0}\left[a\vec{\mathcal{B}} - \left(a - \frac{1}{\gamma^2-1}\right)\left(\vec{\beta} \times \vec{\mathcal{E}}\right)\right] \tag{125}$$

identical to the formula of ref. [12].

This frequency will not depend on the electric field if:

$$a - \frac{1}{\gamma^2-1} = 0 \tag{126}$$

or:

$$\gamma = \sqrt{1 + \frac{1}{a}} = 29.3 \tag{127}$$

The corresponding momentum (the "magic momentum") is 3.09 GeV/c.

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