TRIUMF Experiment E614
Technical Note No.55

Radiative muon decay

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December 4, 2001

1 Introduction

The radiative muon decay:

$$\mu^+ \to e^+ \nu_e \overline{\nu}_\mu \gamma \tag{1}$$

always accompanies the ordinary muon decay:

$$\mu^+ \to e^+ \nu_e \overline{\nu}_\mu \tag{2}$$

Its branching ratio is $(1.4 \pm 0.4)\%$.[1]

This radiative decay is included in the calculation of the radiative corrections[2, 3, 4, 5]. Hence, in the E614 experiment the photon should stay undetected. A problem may arise if the photon converts externally into an e^+-e^- pair. This could lead to some confusion in the interpretation of the event. Rejecting the event would cause a systematic effect. It is therefore important to review the properties of this decay: kinematics, energy and angular distributions. A Monte Carlo calculation will probably be necessary.

This radiative decay has also been studied in the context of $\mu^+ \to e^+\gamma$ searches, where it constitutes an inevitable background in the kinematical region where the two neutrinos carry little energy.

In the following, the kinematics and physics will be discussed and the results of some numerical calculations will be presented.

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2 Kinematics

The muon is supposed to decay from rest. The following notations will be used ($\hbar = c = 1$):

	μ^+	e^+	ν_e	$\overline{\nu}_\mu$	γ
Mass	m_μ	m_e	0	0	0
Momentum	0	$\vec{p}$	$\vec{q}_1$	$\vec{q}_2$	$\vec{k}$
Total energy	m_μ	E	q_1	q_2	k
Kinetic energy	0	T	q_1	q_2	k

2.1 Conservation laws

Conservation of energy-momentum:

$$\begin{aligned} m_\mu &= E + q_1 + q_2 + k \\ 0 &= \vec{p} + \vec{q}_1 + \vec{q}_2 + \vec{k} \end{aligned} \tag{3}$$

It is useful to introduce as a variable the invariant mass m_{12} of the neutrinos:

$$m_{12}^2 = (q_1 + q_2)^2 - (\vec{q}_1 + \vec{q}_2)^2 = (m_\mu - E - k)^2 - (\vec{p} + \vec{k})^2$$

$$m_{12}^2 = (m_\mu - m_e)^2 - 2m_\mu T - 2m_\mu k + 2k(T + m_e)(1 - \beta\cos\theta) \tag{4}$$

where θ is the angle between e^+ and γ, and:

$$\beta = \frac{p}{E} \tag{5}$$

2.2 Dimensionless variables

Some particular configurations are interesting:

1. The positron is created at rest ($E = m_e$, $T = 0$) and the neutrinos are emitted opposite to the photon.

$$m_\mu = m_e + (q_1 + q_2) + k = m_e + 2k \tag{6}$$

   The photon momentum is:

$$k_1 = \frac{m_\mu - m_e}{2} \tag{7}$$

2. The neutrinos carry zero energy ($q_1 = q_2 = 0$) and the positron is emitted opposite to the photon.

$$m_\mu = E + k = \sqrt{m_e^2 + k^2} + k \tag{8}$$

The photon momentum is:

$$k_2 = \frac{m_\mu^2 - m_e^2}{2m_\mu} = \frac{m_\mu - m_e}{2}\,\frac{m_\mu + m_e}{m_\mu} \tag{9}$$

It is obvious that $k_2 > k_1$ and, therefore, k_2 is the maximal energy carried by the photon. It is useful to define a dimensionless γ-ray energy y by:

$$k = \frac{m_\mu^2 - m_e^2}{2m_\mu}\; y = \frac{m_\mu}{2}\; y\; (1 - \epsilon^2) \tag{10}$$

where:

$$\epsilon = \frac{m_e}{m_\mu} = 4.836 \times 10^{-3} \tag{11}$$

This is a small number, but let us keep is throughout the calculation.

In this kinematical configuration the momentum of the positron is:

$$p = \frac{m_\mu^2 - m_e^2}{2m_\mu} = \frac{m_\mu}{2}(1 - \epsilon^2) \tag{12}$$

Its total energy is:

$$E = \frac{m_\mu^2 + m_e^2}{2m_\mu} = \frac{m_\mu}{2}(1 + \epsilon^2) \tag{13}$$

and its kinetic energy:

$$T = \frac{(m_\mu - m_e)^2}{2m_\mu} = \frac{m_\mu}{2}(1 - \epsilon)^2 \tag{14}$$

3. The photon carries zero energy and the neutrinos are emitted opposite to the positron.

$$m_\mu = E + q_1 + q_2 = \sqrt{m_e^2 + p^2} + p \tag{15}$$

One gets for the positron the same result as in the previous configuration. It is therefore useful to define a dimensionless kinetic energy x for the positron:

$$T = \frac{(m_\mu - m_e)^2}{2m_\mu}\; x = \frac{m_\mu}{2}\; x\; (1 - \epsilon)^2 \tag{16}$$

In this kinematical configuration the total energy of the positron is:

$$E = \frac{(m_\mu - m_e)^2}{2m_\mu}\; x\; + m_e = \frac{m_\mu}{2}\left[x(1 - \epsilon)^2 + 2\epsilon\right] \tag{17}$$

and the photon energy:

$$k = \frac{m_\mu}{2} y(1 - \epsilon^2) \tag{18}$$

4. The positron is created at rest, the photon carries zero energy and the neutrinos are emitted back to back.

$$\begin{aligned} m_\mu &= m_e + q_1 + q_2 \\ 0 &= \vec{q}_1 + \vec{q}_2 \end{aligned} \tag{19}$$

The invariant mass m_{12} reaches its maximal value:

$$m_{12} = m_\mu - m_e = m_\mu(1-\epsilon) \tag{20}$$

It is therefore useful to define a dimensionless invariant mass z for the neutrinos:

$$m_{12} = m_\mu \; z \; (1-\epsilon) \tag{21}$$

The new dimensionless variables x, y and z are given by:

$$\begin{aligned} k &= \frac{m_\mu}{2} \; y \; (1-\epsilon^2) \\ T &= \frac{m_\mu}{2} \; x \; (1-\epsilon)^2 \\ E &= \frac{m_\mu}{2} \left[x(1-\epsilon)^2 + 2\epsilon \right] \qquad (22) \\ m_{12} &= m_\mu \; z \; (1-\epsilon) \qquad (23) \end{aligned}$$

Their limits of variation are:

$$0 \leq x \leq 1 \qquad 0 \leq y \leq 1 \qquad 0 \leq z \leq 1 \tag{24}$$

It should be noted that several authors use different definitions for x and y. Some of them neglect ϵ and others define x by using the total energy (including the mass) of the positron. For example, the definitions used by Kuno and Okada[9] are related to the definitions used in this technote:

$$\begin{aligned} x(\text{Kuno}) &= x(1-\epsilon)^2 + 2\epsilon \\ y(\text{Kuno}) &= y(1-\epsilon^2) \end{aligned} \tag{25}$$

They also use a parameter r, related to ϵ:

$$r = \epsilon^2 \tag{26}$$

2.3 Relations between energies and angle

Equation 4 becomes:

$$(1-\epsilon)^2 z^2 = (1-\epsilon)^2(1-x) - (1-\epsilon^2)y\left[1 - \left(\epsilon + \frac{x}{2}(1-\epsilon)^2\right)(1-\beta\cos\theta)\right] \tag{27}$$

which gives:

$$y = \frac{1-\epsilon}{1+\epsilon}\ \frac{1-x-z^2}{1-\Big[\epsilon + (x/2)(1-\epsilon)^2\Big](1-\beta\cos\theta)} \tag{28}$$

The expression of β is

$$\beta = \frac{\sqrt{x(1-\epsilon)^2[x(1-\epsilon)^2+4\epsilon]}}{x(1-\epsilon)^2+2\epsilon} \tag{29}$$

After a few lines of calculation one also gets the expression of x as a function of y.

$$x = \frac{4}{[y(1+\epsilon)(1-\epsilon)-2]^2 - y^2(1+\epsilon)^2(1-\epsilon)^2\cos^2\theta}$$

$$\Big\{[y(1-\epsilon)(1+\epsilon)-2][y(1+\epsilon)-(1-z^2)] + y^2\epsilon(1+\epsilon)^2\cos^2\theta$$

$$\pm y\cos\theta(1+\epsilon)\sqrt{y(1+\epsilon)-(1-z^2)}\ \times$$

$$\sqrt{(1-\epsilon)^2[y(1+\epsilon)-(1-z^2)] + 2\epsilon[y(1-\epsilon)(1+\epsilon)-2] + \epsilon^2 y^2(1+\epsilon)^2\cos^2\theta}\Big\} \tag{30}$$

Sometimes one neglects the positron mass ($\epsilon = 0$, $\beta = 1$). The expression simplifies:

$$y = \frac{1-x-z^2}{1-(x/2)(1-\cos\theta)} = \frac{1-x-z^2}{1-\lambda x} \tag{31}$$

where:

$$\lambda = \sin^2\frac{\theta}{2} \tag{32}$$

Let us note that for $x = 1$:

$$\beta = \frac{\sqrt{(1-\epsilon)^2(1+\epsilon)^2}}{(1+\epsilon^2)} = \frac{1-\epsilon^2}{1+\epsilon^2} \tag{33}$$

and:

$$1+\beta = 1+\frac{(1-\epsilon^2)}{(1+\epsilon^2)} = \frac{2}{(1+\epsilon^2)} \tag{34}$$

What happens for $z = 0$ and $\theta = \pi$?

$$y = \frac{1-\epsilon}{1+\epsilon}\ \frac{1-x}{1-\Big[\epsilon + (x/2)(1-\epsilon)^2\Big](1+\beta)} \tag{35}$$

For $x = 1$ the numerator is equal to zero, but also the denominator:

$$1 - \frac{1}{2}\Big[2\epsilon + (1-\epsilon)^2\Big](1+\beta) = 1 - \frac{1}{2}(1+\epsilon^2)\frac{2}{(1+\epsilon^2)} = 0 \tag{36}$$

Therefore the kinematically allowed domain is bounded by the straight line:

$$x = 1 \qquad 0 \leq y \text{ (any value) } \leq 1 \tag{37}$$

Let us consider the case $z = 0$ and $\theta = 0$ or π and determine the expression of x.

$$y = \frac{1-\epsilon}{1+\epsilon}\,\frac{1-x}{1 - \Big[\epsilon + (x/2)(1-\epsilon)^2\Big](1\pm\beta)} \tag{38}$$

$$1 - \Big[\epsilon + (x/2)(1-\epsilon)^2\Big](1\pm\beta) = \frac{1-\epsilon}{1+\epsilon}\,\frac{1-x}{y} \tag{39}$$

$$\Big[\epsilon + (x/2)(1-\epsilon)^2\Big](1\pm\beta) = 1 - \frac{1-\epsilon}{1+\epsilon}\,\frac{1-x}{y} \tag{40}$$

$$\Big[2\epsilon + x(1-\epsilon)^2\Big](1\pm\beta) = 2\frac{(1+\epsilon)y - (1-\epsilon)(1-x)}{y(1+\epsilon)} \tag{41}$$

and with:

$$\beta = \frac{\sqrt{x(1-\epsilon)^2[4\epsilon + x(1-\epsilon)^2]}}{2\epsilon + x(1-\epsilon)^2} \tag{42}$$

$$2\epsilon + x(1-\epsilon)^2 \pm \sqrt{(1-\epsilon)^4x^2 + 4\epsilon(1-\epsilon)^2x} = 2\frac{(1+\epsilon)y - (1-\epsilon)(1-x)}{y(1+\epsilon)} \tag{43}$$

$$\pm\sqrt{(1-\epsilon)^4x^2 + 4\epsilon(1-\epsilon)^2x} = 2\frac{(1+\epsilon)y - (1-\epsilon)(1-x)}{y(1+\epsilon)} - 2\epsilon - x(1-\epsilon)^2 \tag{44}$$

$$\pm\sqrt{(1-\epsilon)^2x^2 + 4\epsilon x} = \frac{2y(1+\epsilon) - 2(1-x) - xy(1-\epsilon^2)}{y(1+\epsilon)} \tag{45}$$

$$\Big[(1-\epsilon)^2x^2 + 4\epsilon x\Big]y^2(1+\epsilon)^2 = 4y^2(1+\epsilon)^2 + 4(1-x)^2 + x^2y^2(1-\epsilon^2)^2$$

$$-8y(1-x)(1+\epsilon) - 4y^2x(1+\epsilon)^2(1-\epsilon) + 4xy(1-x)(1-\epsilon)^2 \tag{46}$$

$$4y^2(1+\epsilon)^2(1-x)+4(1-x)^2-8y(1+\epsilon)(1-x)+4xy(1-x)(1-\epsilon^2)=0 \tag{47}$$

$$(1-x)\left[y^2(1+\epsilon)^2+(1-x)-2y(1+\epsilon)+xy(1-\epsilon^2)\right]=0 \tag{48}$$

There is a solution $x=1$ and:

$$x=\frac{y^2(1+\epsilon)^2-2y(1+\epsilon)+1}{1-y(1-\epsilon^2)} \tag{49}$$

or:

$$x=\frac{[1-y(1+\epsilon)]^2}{1-y(1-\epsilon^2)} \tag{50}$$

The domain allowed by kinematics is bounded by:

$$x=1 \tag{51}$$

and:

$$x=\frac{[1-y(1+\epsilon)]^2}{1-y(1-\epsilon^2)} \tag{52}$$

Neglecting ϵ the latter equation gives:

$$y=1 \tag{53}$$

and:

$$x=1-y \tag{54}$$

2.4 Figures

Figure 1 shows the relation between positron energy and photon energy for five different values of z^2 (the invariant mass of the neutrinos, squared). For each case six graphs are given; they correspond to six values of $\cos\theta$ (Table 1). Since ϵ is very small these graphs cannot be distinguished from those with $\epsilon=0$.

To see the effect of ϵ, Figures 2 and 3 shows the cases $z^2=0$ and $z^2=0.1$ for six different angles θ and $\epsilon=4.836\times10^{-1}$ (the positron mass is 100 times larger than the normal value).

Graph #	1	2	3	4	5	6
$\cos\theta$	1.0	0.6	0.2	-0.2	-0.6	-1.0
θ (deg.)	0.0	53.0	78.5	101.5	126.9	180.0

Table 1: Graphs obtained for six different angles θ.

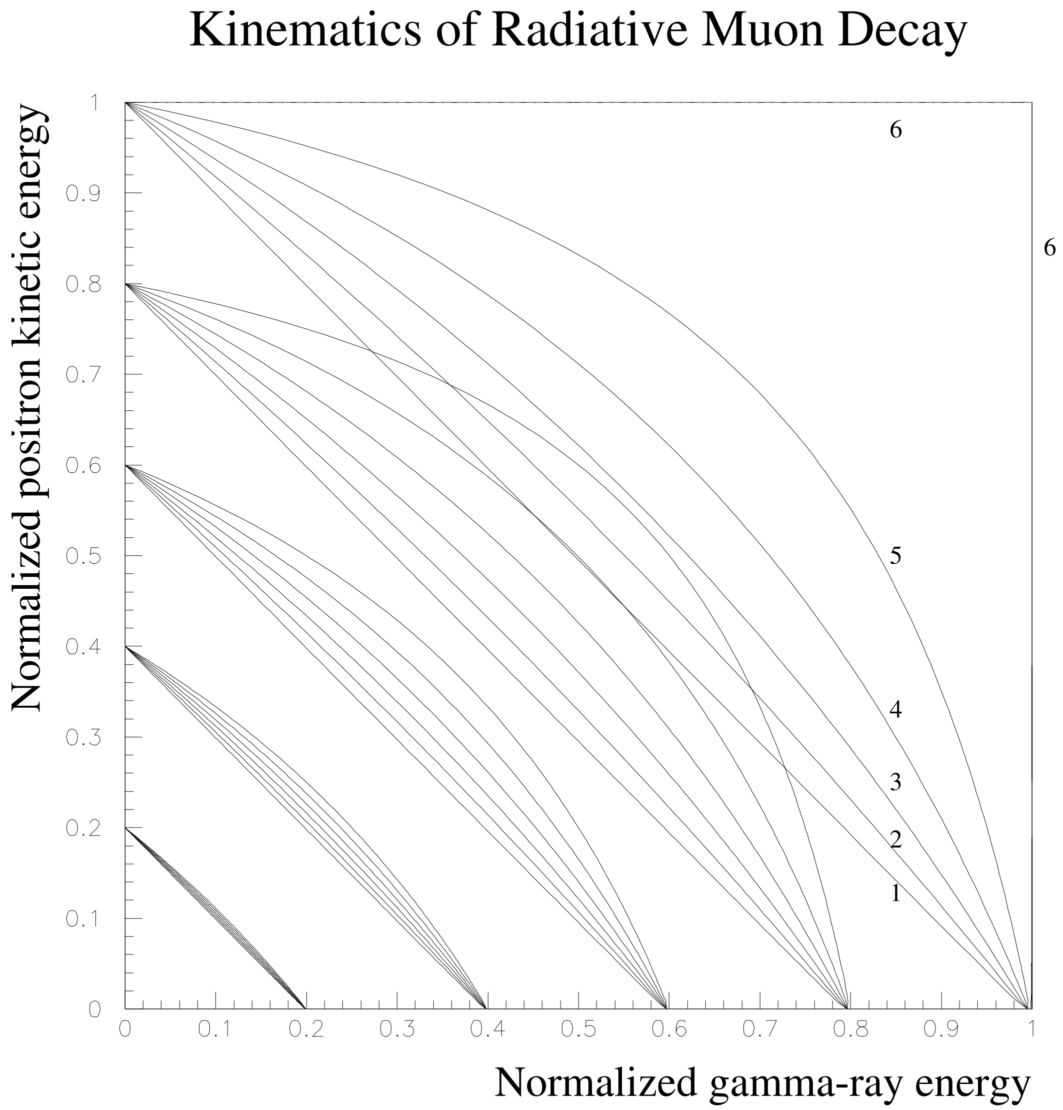

Figure 1: Positron energy vs. gamma-ray energy for five different values of z^2 and six different values of θ (see Table 1). The values of z^2 are 0.0, 0.2, 0.4, 0.6, 0.8 (from top to bottom).

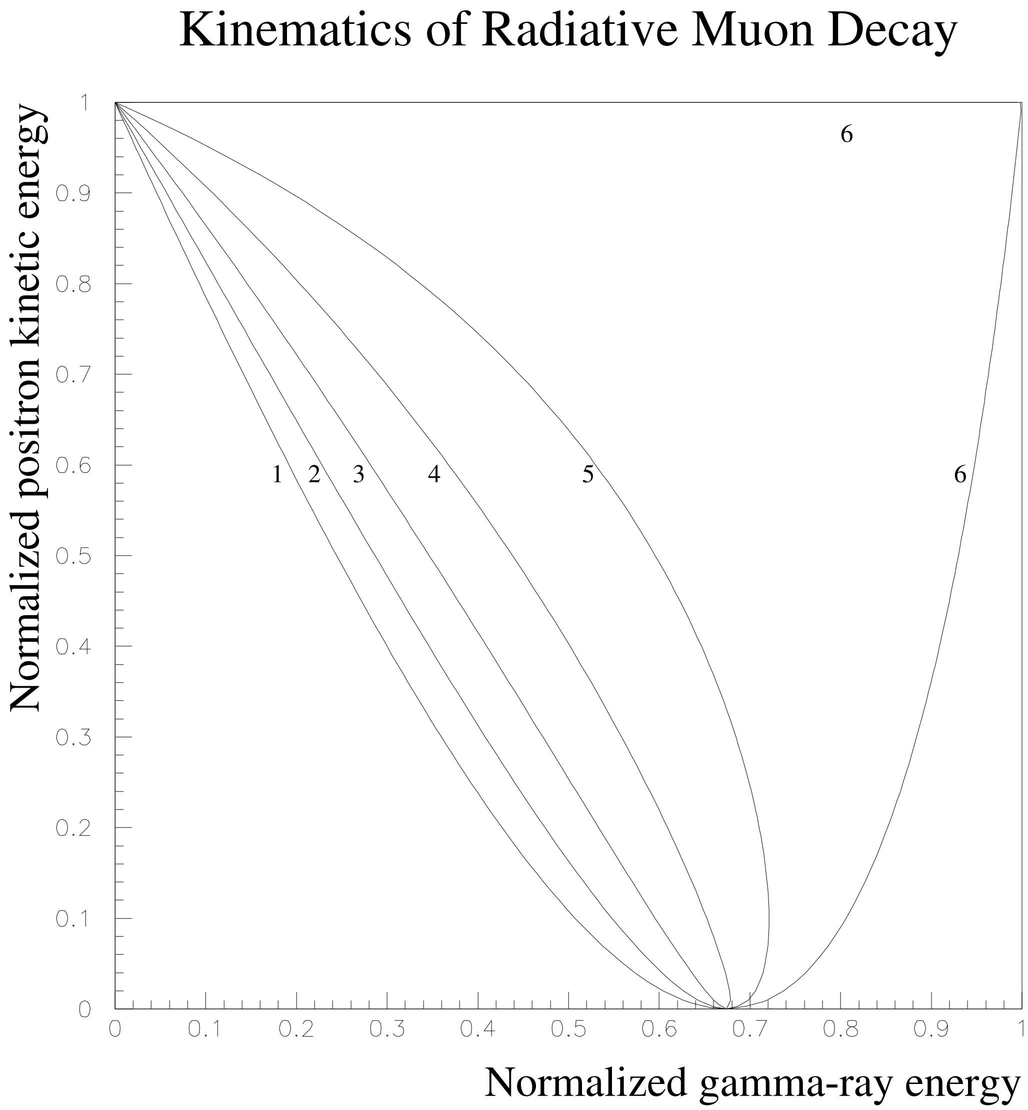

Figure 2: Positron energy vs. gamma-ray energy for $z^2 = 0$ and six different values of θ (see Table 1) and $\epsilon = 4.836 \times 10^{-1}$. Here the positron mass has been increased by a factor of 100 to make the effect of this mass visible.

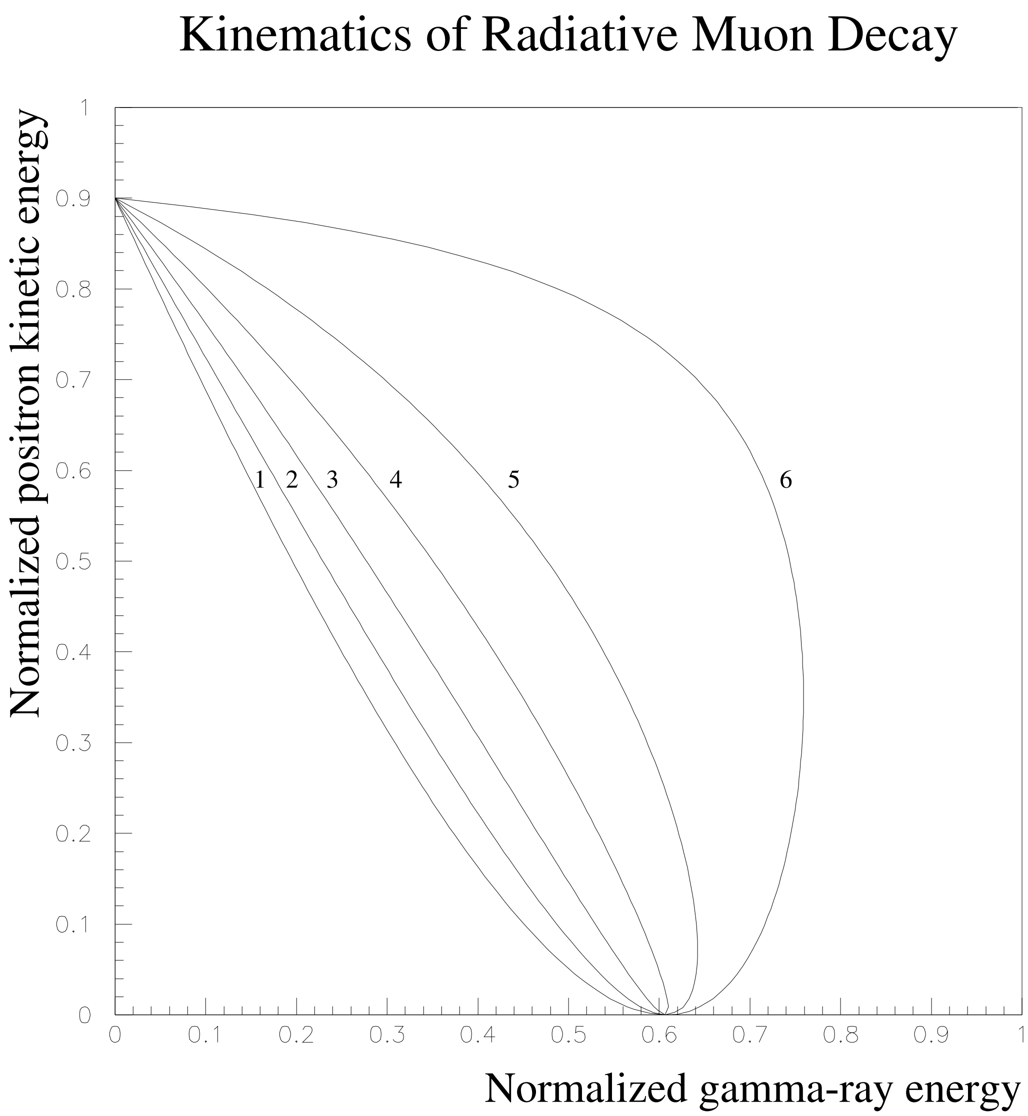

Figure 3: Same as previous graph, but with $z^2 = 0.1$.

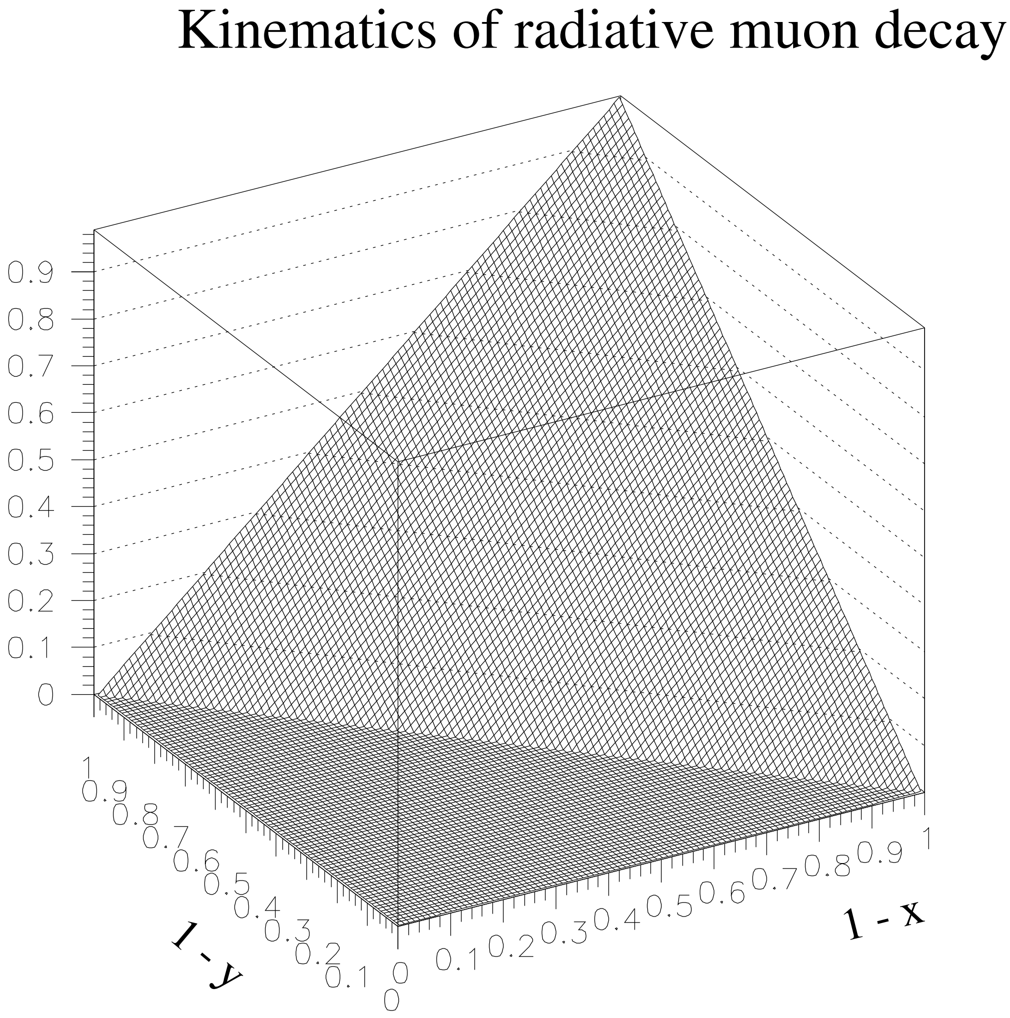

Figure 4: The surface $z^2 = f(x, y)$ for $\theta = 0$

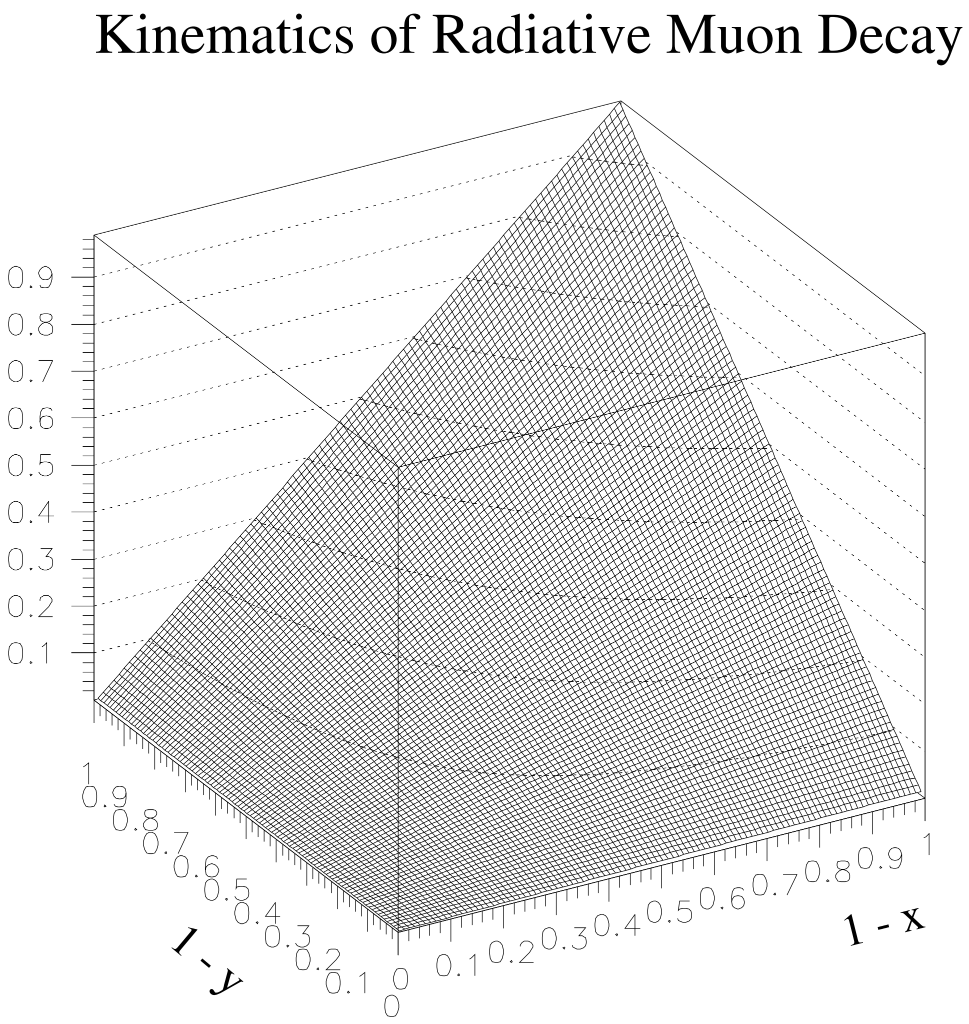

Figure 5: The surface $z^2 = f(x, y)$ for $\theta = \pi$

3 Decay rate

The decay rate of the radiative muon decay has been calculated many times. Fronsdal and Überall[6] give references to earlier papers. In this report we will consider the calculations by Fronsdal and Überall[6], Kalyniak and Ng[3, 4], Brunet[7, 8], Kuno and Okada[9]. The most complete calculation is the one by Fronsdal and Überall. They calculate the rates for both the non-radiative and the radiative decay, including the muon polarization. The hamiltonian includes all five interactions, scalar, vector, tensor, axial-vector and pseudoscalar. The electron (positron) mass is neglected except in those terms which would diverge in certain kinematical configurations.

3.1 The non-radiative decay

The non-radiative decay rate is often used as a normalization for the radiative decay, therefore is it important to understand the notations used by different authors.

The non-radiative decay rate (not including the effect of the radiative corrections) is given by[6]:

$$dN = \frac{\mu^5}{3 \times 2^{10}\pi^4} x^2 dx d\Omega_e \omega \left\{ 3(1-x) + 2\rho\left(\frac{4}{3}x - 1\right) \mp P s_e \xi \left[1 - x + 2\delta\left(\frac{4}{3}x - 1\right)\right]\right\} \tag{55}$$

In this expression the parameter η does not appear because the electron (positron) mass is set to zero. The variable x is defined as:

$$x = \frac{E_e}{(E_e)_{max}} = \frac{2E_e}{\mu} \tag{56}$$

where μ is the muon mass.

P is the degree of polarization of the muon ensemble ($0 \leq P \leq 1$).

The $\mp$ sign in front of the polarization has to be interpreted in the following way:

$$\begin{array}{lll} + & \text{for} & \mu^+\text{decay} \\ - & \text{for} & \mu^-\text{decay} \end{array} \tag{57}$$

$s_e = \cos(\vec{s}, \vec{p}_e)$, where $\vec{s}$ is a unit vector along the muon polarization vector and $\vec{p}_e$ is a unit vector along the electron (positron) momentum.

The value of ω is related to the lifetime of the muon. Summing over the polarization states give:

$$dN = \frac{\mu^5}{3 \times 2^{10}\pi^4} x^2 dx d\Omega_e \omega \left[3(1-x) + 2\rho\left(\frac{4}{3}x - 1\right)\right] \tag{58}$$

Integrating over x (from 0 to 1) and over Ω_e (4π) gives:

$$N = \frac{\mu^5}{3 \times 2^{10}\pi^3}\omega \tag{59}$$

This can be identified with the expression for the muon lifetime in the $V-A$ theory, neglecting corrections:

$$\frac{1}{\tau_\mu} = \frac{\mu^5 G_F^2}{3 \times 2^6 \pi^3} \tag{60}$$

Hence:

$$\omega = 16 G_F^2 \tag{61}$$

In fact the definition of ω is:

$$\omega = |C_S|^2 + |C'_S|^2 + |C_P|^2 + |C'_P|^2 + 4\Bigg(|C_V|^2 + |C'_V|^2 + |C_A|^2 + |C'_A|^2\Bigg) \tag{62}$$

$$+6\Bigg(|C_T|^2 + |C'_T|^2\Bigg) \tag{63}$$

In the $V-A$ theory:

$$C_S = C'_S = C_P = C'_P = C_T = C'_T = 0 \tag{64}$$

$$C_A = C'_A = -C_V = -C'_V \tag{65}$$

Hence:

$$\omega = 16|C_V|^2 \tag{66}$$

The paper also allows to calculate the rates for emission of longitudinally polarized electrons (positrons). This will not be used in the present report.

3.2 The radiative decay

3.2.1 Fronsdal and Überall

The rate for the radiative decay is given by[6]:

$$dN_\gamma = e^2 \frac{\mu^5}{3 \times 2^{16} \pi^6} dx \frac{dy}{y} d\Omega_e d\Omega_\gamma Z \tag{67}$$

Here e^2 stands for $\alpha = 1/137$.

The branching ratio for the radiative decay is:

$$dB_\gamma = \frac{e^2}{64\pi^3} dx \frac{dy}{y} d\Omega_e d\Omega_\gamma \frac{Z}{\omega} \tag{68}$$

The variable y is defined as:

$$y = \frac{E_\gamma}{(E_\gamma)_{max}} = \frac{2E_\gamma}{\mu} \tag{69}$$

and the quantity Z is given by:

$$Z = \omega\Bigg\{n_V + \left(1 - \frac{4}{3}\rho\right)\left(2n_S + n_V - n_T\right) + \eta\left(2n_S - 2n_V + n_T\right)$$

$$\pm P\xi \sum_{k=e,\gamma} s_k \left[n_V^k - \frac{1}{3}\left(1 - \frac{4}{3}\delta\right)\left(2n_S^k + 5n_V^k - n_T^k\right) + \kappa\left(2n_S^k - 2n_V^k + n_T^k\right)\right]\Bigg\} \qquad (70)$$

where κ is a new parameter, defined as:

$$\kappa\xi\omega = 2\mathcal{R}e(C_S^* C_S' + C_P^* C_P') + 4\mathcal{R}e(C_T^* C_T') \qquad (71)$$

The $\pm$ sign must be interpreted as:

$$\begin{array}{lll} + & \text{for} & \mu^-\text{decay} \\ - & \text{for} & \mu^+\text{decay} \end{array} \qquad (72)$$

(note the difference in sign when comparing to the non-radiative decay).

The terms $n_i^{(k)}$ are defined as:

$$n_i^{(k)} = \left[\Delta + \frac{2m_e^2}{m_\mu^2 x^2}\right]^{-1} F_{-1}^{i(k)} + F_0^{i(k)} + \Delta F_1^{i(k)} + \Delta^2 F_2^{i(k)} \qquad (73)$$

Note that the authors have neglected the positron mass. This leads to $\beta = 1$ and to a divergence of the term Δ^{-1} for $\theta = 0$. This has been avoided by adding a correction to this term (see Appendix 1).

In the above expression $i = S, V, T$. The other interactions A and P are, however, taken into account by using the projector $(1 + \gamma^5)$.

The expressions for the functions $F_j^{i(k)}$ are given in Appendix 2.

In the Standard Model the expression simplifies to:

$$Z = \omega\left[n_V \pm P \sum_{k=e,\gamma} s_k n_V^k\right] \qquad (74)$$

with the following values of the parameters:

$$\begin{array}{rcl} \rho & = & 0.75 \\ \eta & = & 0.00 \\ \xi & = & 1.00 \\ \delta & = & 0.75 \\ \kappa & = & 0.00 \end{array} \qquad (75)$$

Angular distributions for the photon are shown in Figures 6 to 11.

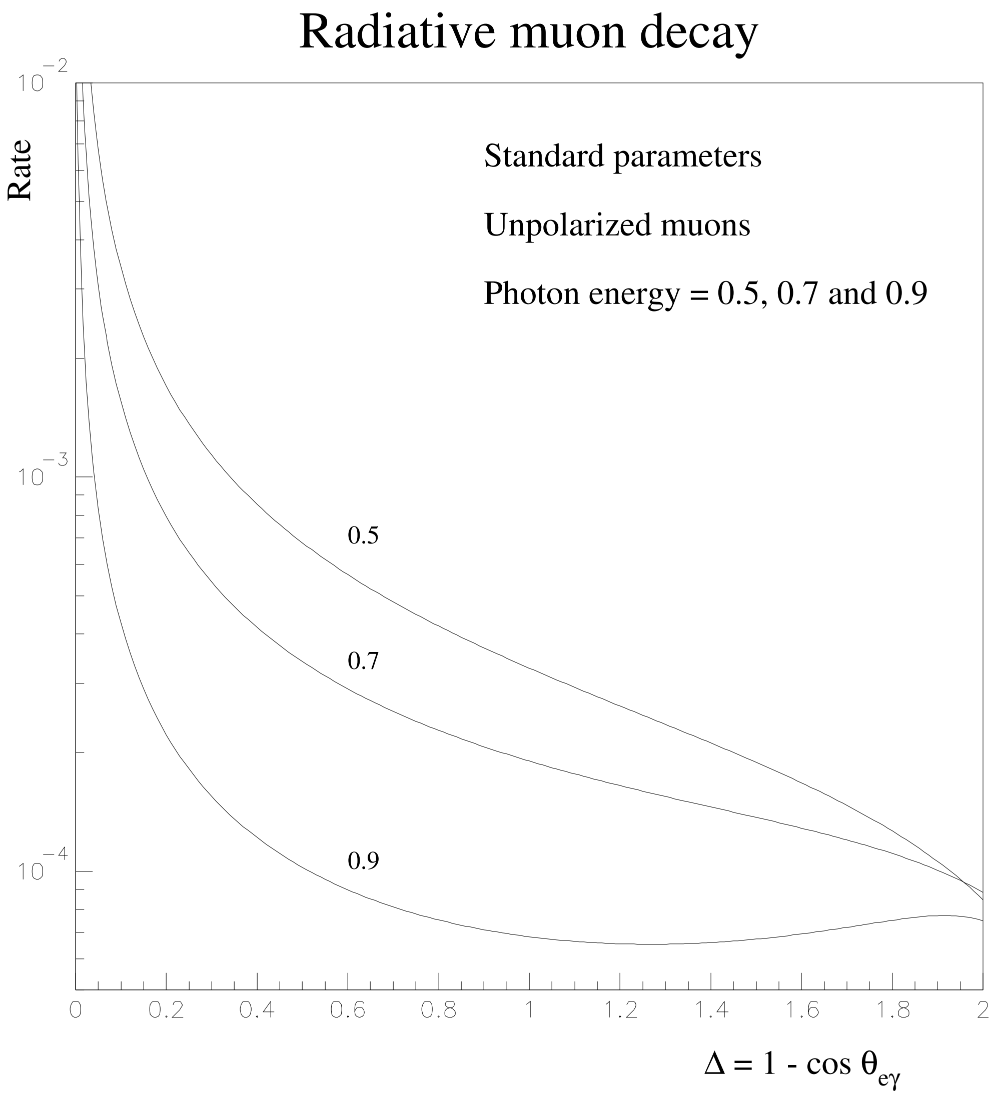

Figure 6: This graph reproduces the one given in the paper by Fronsdal and Überall (their Figure 1). Angular distribution of the photons as a function of Δ for three values of the photon energy, unpolarized muons, after integration over positron energies, and normalized to the non-radiative rate.

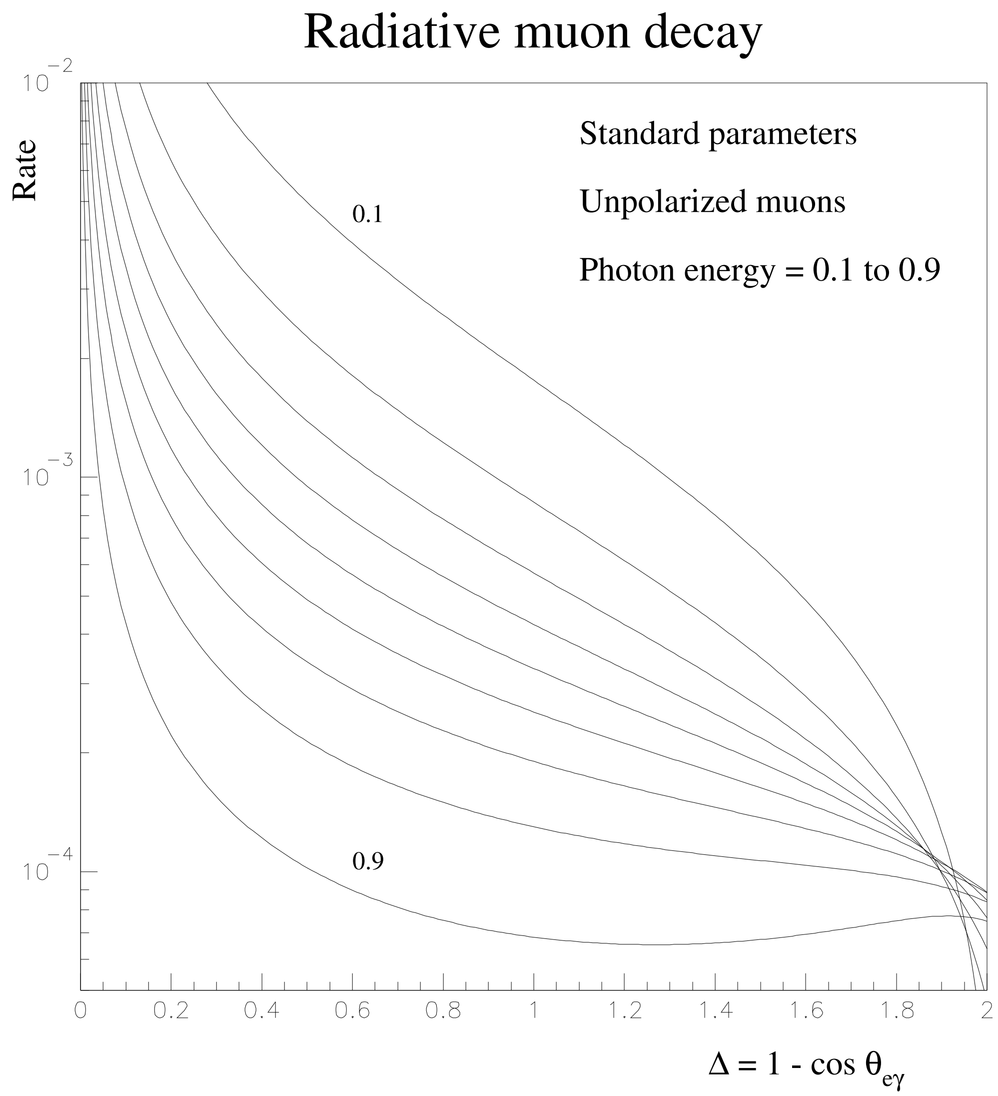

Figure 7: Same as the previous graph, but for nine different photon energies.

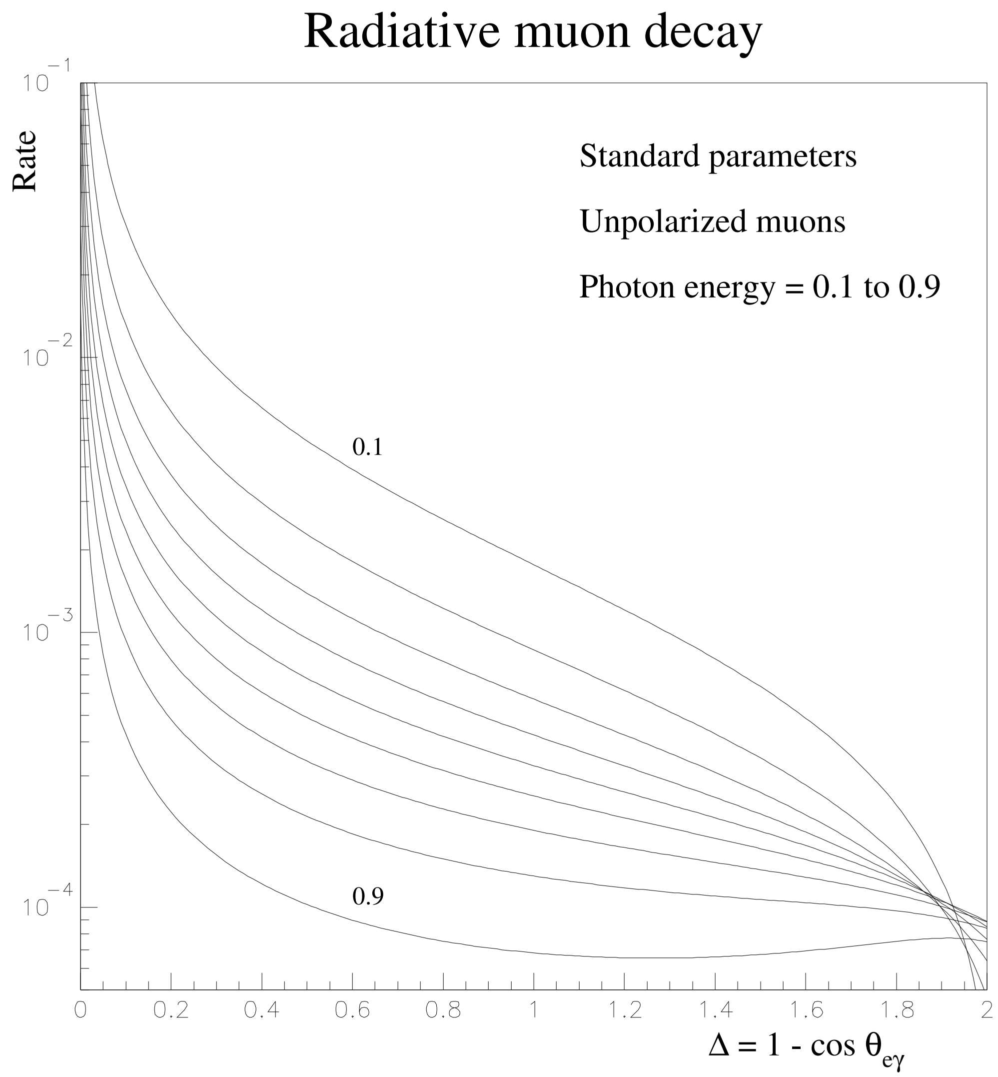

Figure 8: Same as the previous graph, but with a different vertical scale.

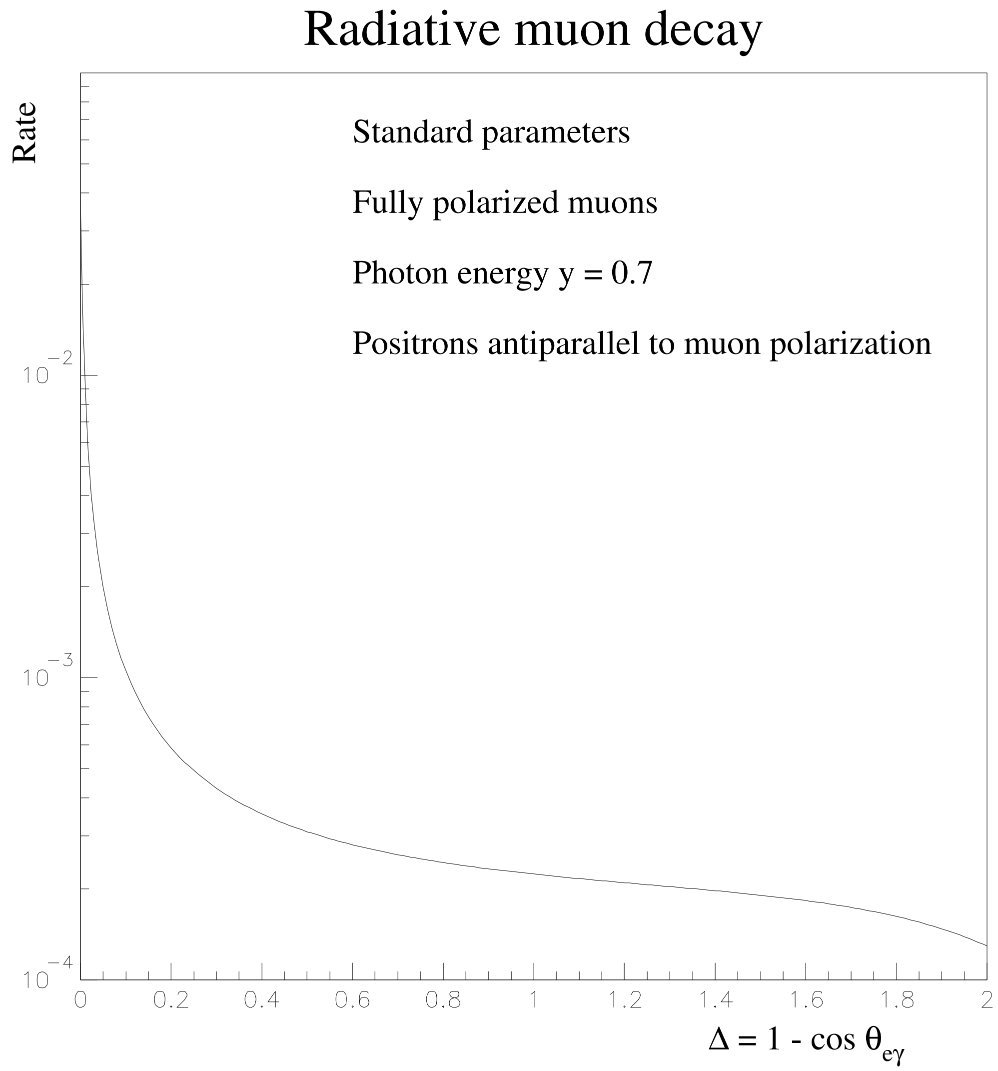

Figure 9: This graph reproduces the one given in the paper by Fronsdal and Überall (their Figure 2). Angular distribution of the photons as a function of Δ, for a photon energy y = 0.7, fully polarized muons, after integration over positron energies, and normalized to the non-radiative rate.

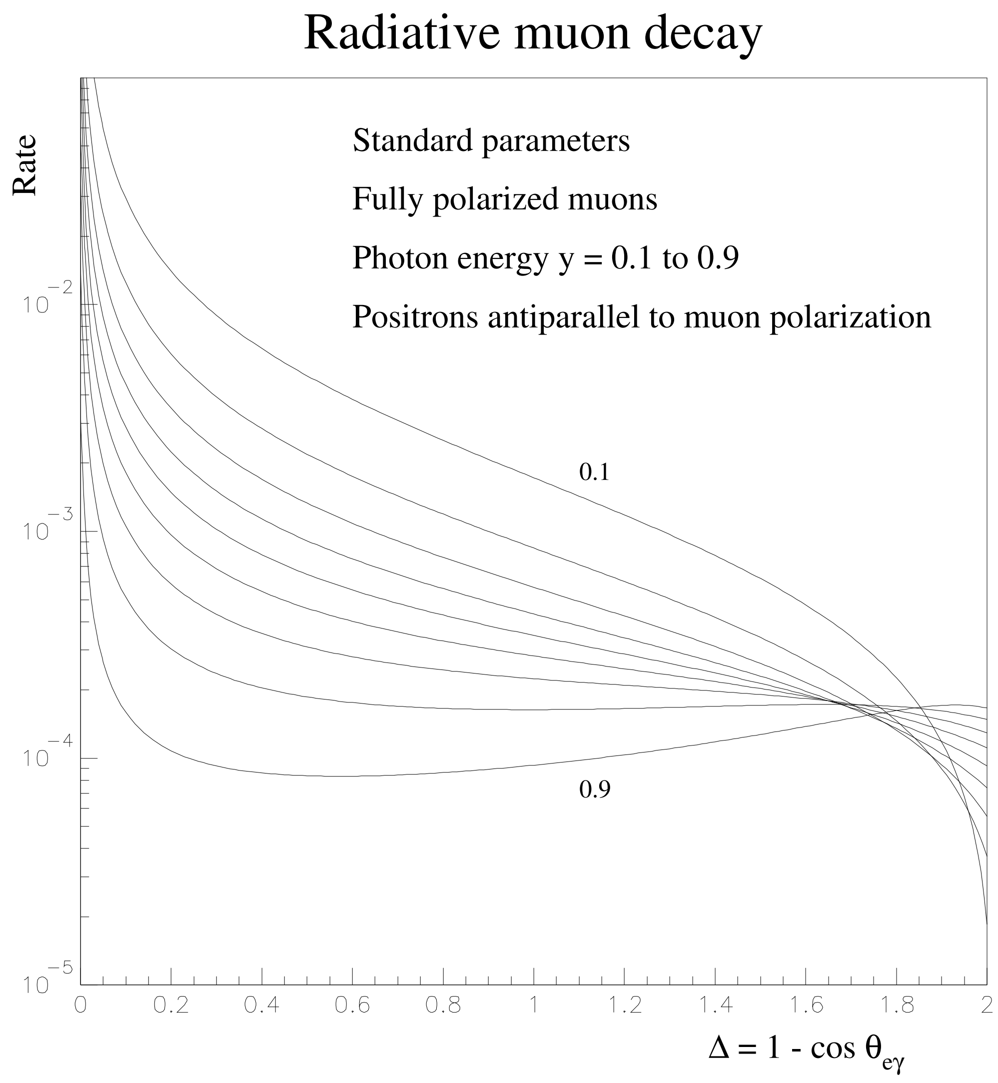

Figure 10: Same as the previous graph, but with nine different photon energies.

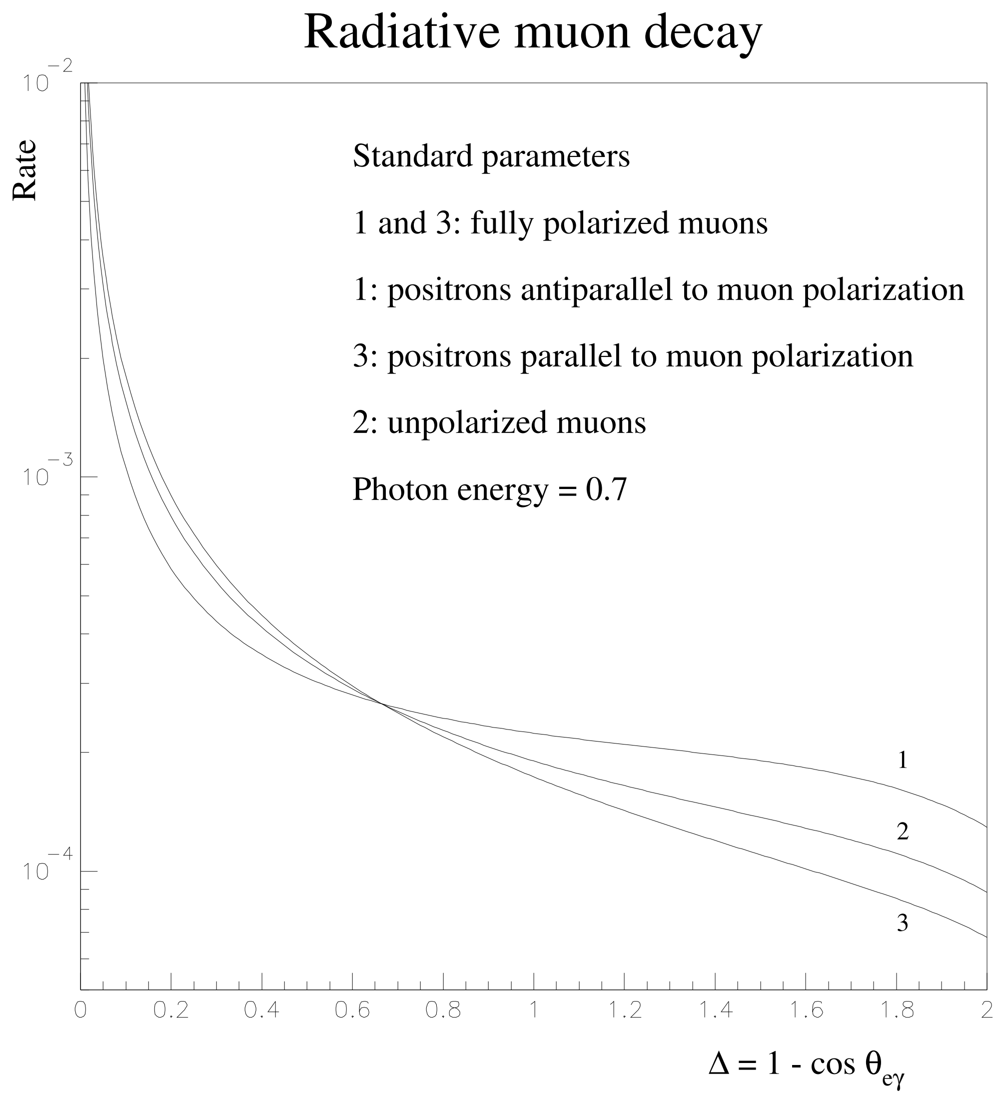

Figure 11: Photon energy 0.7 and three different cases 1, 2 and 3 (see above).

The three curves in Figure 11 have a common point of intersection. This can be checked easily. With the Standard parameters the expression for the radiative decay rate is given by (omitting the constant factor ω):

$$Z = n_V - P(s_e n_V^e + s_\gamma n_V^\gamma) \tag{76}$$

and for the nonradiative decay (after integration over x from 0 to 1):

$$Z_0 = \frac{1}{4}\left[1 + \frac{1}{3}Ps_e\right] \tag{77}$$

We have for the three cases:

1. $P = 1$, $s_e = -1$, $s_\gamma = -\cos\theta_{e\gamma} = \Delta - 1$

$$\frac{Z_1}{4Z_0} = \frac{3}{2}\left[n_V + n_V^e - (\Delta - 1)n_V^\gamma\right] \tag{78}$$

2. $P = 0$

$$\frac{Z_1}{4Z_0} = n_V \tag{79}$$

3. $P = 1$, $s_e = +1$, $s_\gamma = \cos\theta_e\gamma = 1 - \Delta$

$$\frac{Z_1}{4Z_0} = \frac{3}{4}\left[n_V - n_V^e - (1 - \Delta)n_V^\gamma\right] \tag{80}$$

One has to solve two equations:

$$\frac{3}{2}\left[n_V + n_V^e + (1 - \Delta)n_V^\gamma\right] = n_V = \frac{3}{4}\left[n_V - n_V^e - (1 - \Delta)n_V^\gamma\right] \tag{81}$$

These two equations have a common solution:

$$\Delta = \frac{n_V + 3n_V^e + 3n_V^\gamma}{3n_V^\gamma} \tag{82}$$

A numerical calculation gives:

$$\Delta = 0.664 \tag{83}$$

3.2.2 Brunet

The calculation by Brunet[7, 8] is restricted to the $V - A$ theory. The purpose of this calculation was to study the sensitivity of the radiative muon decay to the mass of the intermediate boson W. The rate of the radiative muon decay is expressed as:

$$d\Gamma_\gamma = \frac{1}{\tau_\mu}\frac{8\alpha}{\pi}\left[N_1 + bN_2 + b\kappa N_3 + b^2 qN_4\right]\epsilon d\epsilon\ \omega d\omega\ d(\cos\theta) \tag{84}$$

where $\alpha = 1/137$ and τ_μ is the muon lifetime.

The parameter b expresses the effect of the intermediate boson mass: $b = (m_\mu/m_W)^2$. In the following we will set $b = 0$. Then it is possible to compare with Fronsdal and Überall.

The positron mass is neglected. The definitions of ϵ and ω are:

$$\epsilon = \frac{E_e}{m_\mu} \qquad 0 \le \epsilon \le \frac{1}{2} \tag{85}$$

$$\omega = \frac{E_\gamma}{m_\mu} \qquad 0 \le \omega \le \frac{1}{2} \tag{86}$$

3.2.3 Kuno and Okada

Kuno and Okada[9] have calculated the radiative muon decay rate, assuming the Standard Model and keeping terms in $(m_e/m_\mu)^2$ and $(m_e/m_\mu)^4$. The branching ratio is given by:

$$dB(\mu^\pm \to e^\pm \nu\overline{\nu}\gamma) =$$

$$\frac{\alpha}{64\pi^3}\beta dx \frac{dy}{y} d\Omega_e d\Omega_\gamma \left[F(x,y,d) \mp \beta \vec{P}_\mu \cdot \hat{p}_e G(x,y,d) \mp \vec{P}_\mu \cdot \hat{p}_\gamma H(x,y,d) \right] \tag{87}$$

where $\alpha = 1/137$, $\vec{P}_\mu$ is the muon polarization vector $(0 \le |\vec{P}_\mu| \le 1)$, $\hat{p}_e$ and $\hat{p}_\gamma$ unit vectors along the positron and photon momenta, respectively. And $\beta = |\vec{p}_e|/E_e$.

The definitions of x, y and d are the following:

$$x = \frac{2E_e}{m_\mu} \qquad 0 \le x \le 1 \tag{88}$$

$$y = \frac{2E_\gamma}{m_\mu} \qquad 0 \le y \le 1 \tag{89}$$

$$d = 1 - \beta \hat{p}_e \cdot \hat{p}_\gamma \qquad 1 - \beta \le d \le 1 + \beta \tag{90}$$

The functions F, G and H are developped in powers of $r = (m_e/m_\mu)^2$:

$$\begin{aligned} F &= F^{(0)} + rF^{(1)} + r^2F^{(2)} \\ G &= G^{(0)} + rG^{(1)} + r^2G^{(2)} \\ H &= H^{(0)} + rH^{(1)} + r^2H^{(2)} \end{aligned} \tag{91}$$

The functions $F^{(0)}$, $G^{(0)}$ and $H^{(0)}$ are identical to the corresponding expressions of Fronsdal and Überall:

$$\begin{aligned} F^{(0)} &\equiv n_V \\ G^{(0)} &\equiv n_V^e \\ H^{(0)} &\equiv n_V^\gamma \end{aligned} \tag{92}$$

Kuno and Okada also give formulae for the photon spectrum and angular distribution with respect to the muon polarization vector, after integration over the positron energy and the angle between positron and photon:

$$\frac{dB(\mu^\pm \to e^\pm \nu\overline{\nu}\gamma)}{dy d\cos\theta_\gamma} = \frac{1}{y}\Big[J_+(y)(1 \pm P_\mu \cos\theta_\gamma) + J_-(y)(1 \mp P_\mu \cos\theta_\gamma)\Big] \tag{93}$$

where y is the normalized energy, θ_γ the angle between the photon momentum and the muon polarization vector, and:

$$J_+(y) = \frac{\alpha}{6\pi}(1-y)\Bigg[\Bigg(3F(y) - \frac{17}{2}\Bigg) + \Bigg(-3F(y) + 7\Bigg)(1-y) + \Bigg(2F(y) - \frac{13}{3}\Bigg)(1-y)^2\Bigg]$$

$$J_-(y) = \frac{\alpha}{6\pi}(1-y)^2\Bigg[\Bigg(3F(y) - \frac{93}{12}\Bigg) + \Bigg(-4F(y) + \frac{29}{3}\Bigg)(1-y) + \Bigg(2F(y) - \frac{55}{12}\Bigg)(1-y)^2\Bigg]$$

where:

$$F(y) = \ln\frac{1-y}{r} \tag{94}$$

and:

$$r = \frac{m_e^2}{m_\mu^2} \tag{95}$$

Photon spectra are shown on Figure 12 for unpolarized muons and on Figure 13 for fully polarized muons and three different values for the angle between photon momentum and muon polarization vector.

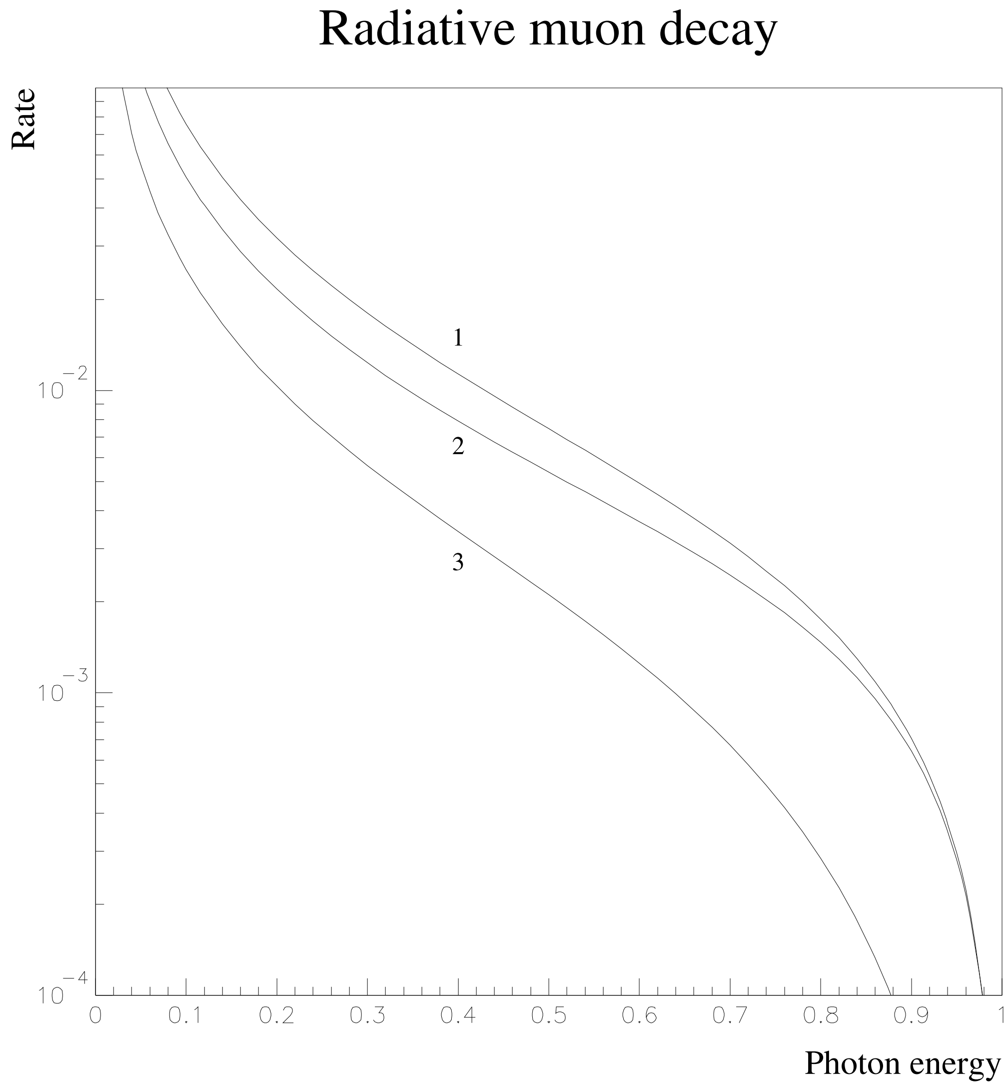

Figure 12: Photon spectrum for unpolarized muons. Graph 2 shows the contribution of the J_+ term, graph 3 the contribution of the J_- term and graph 1 the sum of the two contributions.

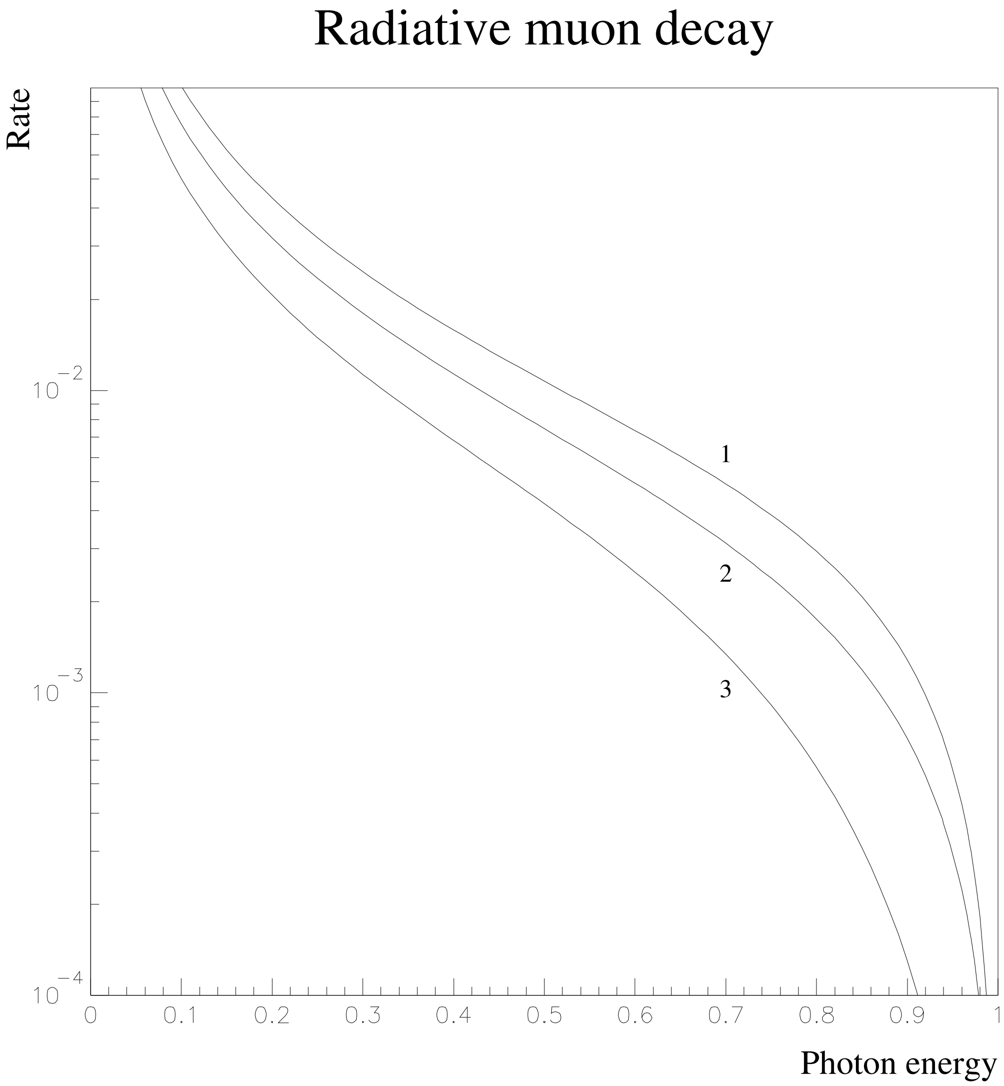

Figure 13: Photon spectrum for fully polarized muons. For three different values of $\cos\theta_\gamma$, θ_γ being the angle between the photon momentum and the muon polarization vector. Graph 1 corresponds to $\cos\theta_\gamma = -1$, graph 2 to $\cos\theta_\gamma = 0$ and graph 3 to $\cos\theta_\gamma = +1$.

3.2.4 Kalyniak and Ng

Kalyniak and Ng[3, 4] have calculated the radiative corrections to the non-radiative decay and the radiative decay with massive neutrinos. Their formulae can be used for massless neutrinos by setting the neutrino masses to zero.

3.2.5 The RADMU ntuple

The most complete information on the radiative muon decay can be stored in an ntuple RADMU obtained in the following way:

1. The muon polarization vector is conventionnaly set antiparallel to the Z axis.

2. A random vector $\vec{V}$ $(0 \leq |\vec{V}| \leq 1)$ is generated for the positron momentum (components V_x, V_y, V_z). The polar and azimutal angles θ_e, ϕ_e, and the normalized energy x are calculated.

3. A random vector $\vec{V}$ $(0 \leq |\vec{V}| \leq 1)$ is generated for the photon momentum (components V_x, V_y, V_z). The polar and azimutal angles θ_γ, ϕ_γ, and the normalized energy y are calculated.

4. The angle $\theta_{e\gamma}$ between the positron and the photon is calculated:

$$\cos\theta_{e\gamma} = \cos\theta_e \cos\theta_\gamma + \sin\theta_e \sin\theta_\gamma \cos\theta(\phi_e - \phi_\gamma) \tag{96}$$

5. The kinematical condition is checked:

$$0 \leq z^2 \leq 1 \tag{97}$$

   and the events which do not satisfy this condition are rejected.

6. The decay rate is calculated according to several authors: Fronsdal and Überall, Kuno and Okada, Kalyniak and Ng.

7. The following variables are stored in the ntuple:

   - x: the normalized positron energy.
   - y: the normalized photon energy.
   - $\cos\theta_e$: the cosine of the angle between the positron momentum and the polarization vector of the muon.
   - $\cos\theta_\gamma$: the cosine of the angle between the photon momentum and the polarization vector of the muon.
   - $\cos\theta_{e\gamma}$: the cosine of the angle between the positron and photon momenta.
   - z^2: the invariant mass of the neutrinos, squared.
   - s: the decay rate.

The next figures show some graphs obtained from the RADMU ntuple.

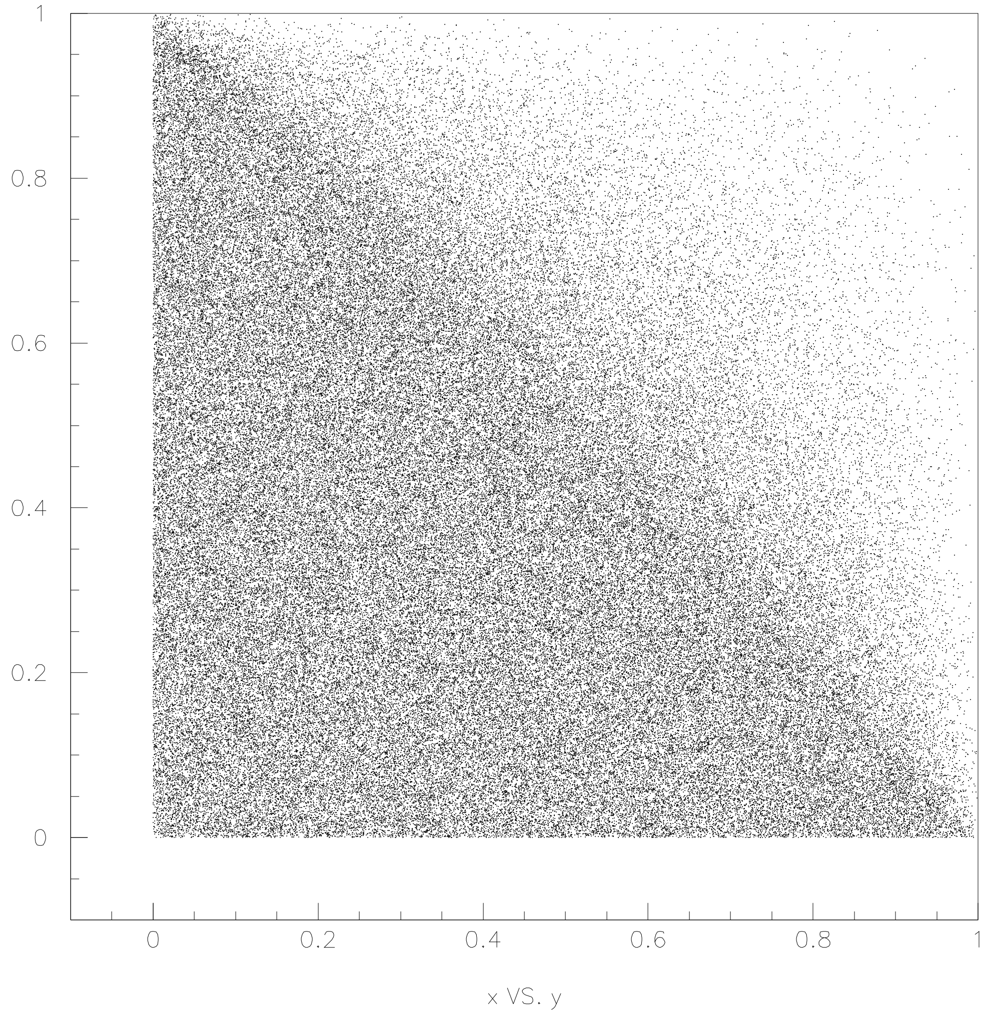

Figure 14: Distribution of the radiative muon decay in the plane x-y. On the vertical axis: x is the positron energy. On the horizontal axis: y is the photon energy.

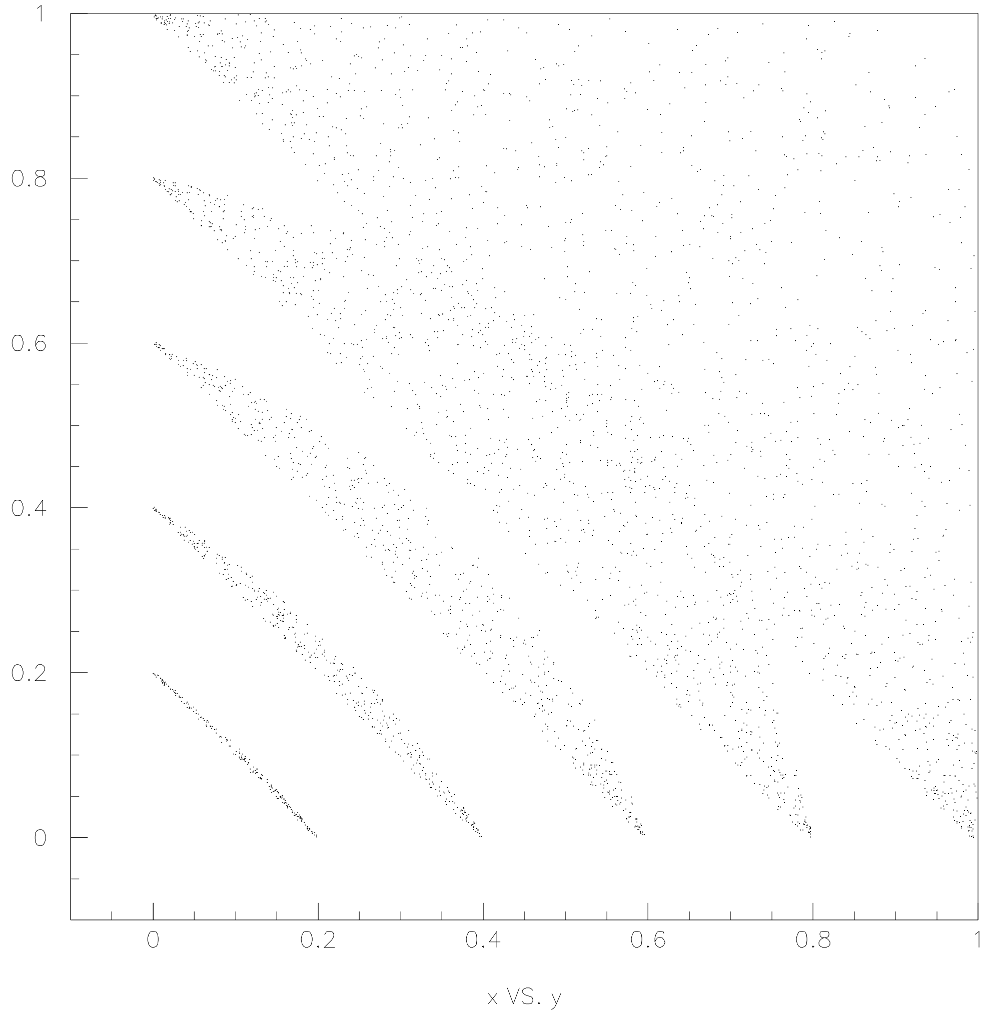

Figure 15: Same as the previous graph but for several values of the invariant mass of the neutrinos: 0.0, 0.2, 0.4, 0.6 and 0.8. To be compared with Figure 1.

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4 Appendix 1: concerning Fronsdal's formula

In the paper by Fronsdal and Überall (Phys. Rev. **113**, 654, 1959) formula (4b), page 655, should be corrected. The term $\mu^2 x^2$ has to be dimensionless. It should be μ^2/m^2x^2, where m is the positron mass and μ the muon mass.

Why is this term necessary?

In the paper the positron mass is neglected, but this approximation leads to a divergence when $\Delta \to 0$ (for $\theta_{e\gamma} \to 0$). In this case the positron mass must be taken into account.

Consider the expression:

$$E_e E_\gamma - \vec{p}_e \cdot \vec{q} = E_e q - p_e q \cos(\vec{p}_e, \vec{q}) \tag{98}$$

We have:

$$E_e = \sqrt{p_e^2 + m_e^2} \approx p_e \left(1 + \frac{1}{2}\frac{m_e^2}{p_e^2}\right) \tag{99}$$

This gives:

$$p_e q \left(1 + \frac{1}{2}\frac{m_e^2}{p_e^2} - \cos(\vec{p}_e, \vec{q})\right) = p_e q \left(\Delta_0 + \frac{1}{2}\frac{m_e^2}{p_e^2}\right) \tag{100}$$

where Δ_0 is the expression of Δ when the approximation $m_e = 0$ is made:

$$\Delta_0 = 1 - \cos(\vec{p}_e, \vec{q}) \tag{101}$$

Using the dimensionless variables x and y:

$$p_e = \frac{m_\mu}{2} x \tag{102}$$

and:

$$q = \frac{m_\mu}{2} y \tag{103}$$

one gets:

$$\Delta = \Delta_0 + 2\frac{m_e^2}{m_\mu^2}\frac{1}{x^2} \tag{104}$$

For $\Delta_0 = 0$ ($\theta_{e\gamma} = 0$):

$$\Delta = 2\frac{m_e^2}{m_\mu^2}\frac{1}{x^2} \tag{105}$$

and there is no divergence, except for $y = 0$ as expected.

For $x = 1$ the correction is very small:

$$\Delta = 2\frac{m_e^2}{m_\mu^2} \approx 4,7 \times 10^{-5} \tag{106}$$

The variable x cannot take the zero value. In fact x should be defined as:

$$x = \frac{2E_e}{m_\mu} \tag{107}$$

and the minimal value is:

$$x = \frac{2m_e}{m_\mu} \tag{108}$$

Here the correction is large:

$$\Delta = \Delta_0 + \frac{1}{2} \tag{109}$$

One should avoid going too close to $\Delta = 0$.

5 Appendix 2: the formulae of Fronsdal and Überall

5.1 Scalar (pseudoscalar) interaction

$$\begin{aligned}
F^S_{-1} &= 12\Big[y^2(1-y) + xy(2-3y) + 2x^2(1-2y) - 2x^3\Big] \\
F^S_0 &= 6\Big[-xy(2-3y^2) - 2x^2(1-y-3y^2) + 2x^3(1+2y)\Big] \\
F^S_1 &= 3\Big[x^2y(2-3y-3y^2) - x^3y(4+3y)\Big] \\
F^S_2 &= \frac{3}{2}x^3y^2(2+y) \\
F^{Se}_{-1} &= 12\Big[xy(1-y) + x^2(2-3y) - 2x^3\Big] \\
F^{Se}_0 &= 6\Big[-x^2(2-y-2y^2) + x^3(2+3y)\Big] \\
F^{Se}_1 &= -3x^3y(2+y) \\
F^{Se}_2 &= 0 \\
F^{S\gamma}_{-1} &= 12\Big[y^2(1-y) + xy(1-2y) - x^2y\Big] \\
F^{S\gamma}_0 &= 6\Big[-xy^2(2-3y) - x^2y(1-4y) + x^3y\Big] \\
F^{S\gamma}_1 &= 3\Big[x^2y^2(1-3y) - 2x^3y^2\Big] \\
F^{S\gamma}_2 &= \frac{3}{2}x^3y^3
\end{aligned}$$

5.2 Vector (pseudovector) interaction

$$
\begin{aligned}
F_{-1}^{V} &= 8\Big[y^2(3-2y) + 6xy(1-y) + 2x^2(3-4y) - 4x^3\Big] \\
F_{0}^{V} &= 8\Big[-xy(3-y-y^2) - x^2(3-y-4y^2) + 2x^3(1+2y)\Big] \\
F_{1}^{V} &= 2\Big[x^2y(6-5y-2y^2) - 2x^3y(4+3y)\Big] \\
F_{2}^{V} &= 2x^3y^2(2+y) \\
F_{-1}^{Ve} &= 8\Big[xy(1-2y) + 2x^2(1-3y) - 4x^3\Big] \\
F_{0}^{Ve} &= 4\Big[-x^2(2-3y-4y^2) + 2x^3(2+3y)\Big] \\
F_{1}^{Ve} &= -4x^3y(2+y) \\
F_{2}^{Ve} &= 0 \\
F_{-1}^{V\gamma} &= 8\Big[y^2(1-2y) + xy(1-4y) - 2x^2y\Big] \\
F_{0}^{V\gamma} &= 4\Big[2xy^2(1+y) - x^2y(1-4y) + 2x^3y\Big] \\
F_{1}^{V\gamma} &= 2\Big[x^2y^2(1-2y) - 4x^3y^2\Big] \\
F_{2}^{V\gamma} &= 2x^3y^3
\end{aligned}
$$

5.3 Tensor interaction

$$
\begin{aligned}
F^{T}_{-1} &= 8\Big[y^2(3-y) + 3xy(2-y) + 2x^2(3-2y) - 2x^3\Big] \\
F^{T}_{0} &= 4\Big[-xy(6+y^2) - 2x^2(3+y-3y^2) + 2x^3(1+2y)\Big] \\
F^{T}_{1} &= 2\Big[x^2y(6-5y+y^2) - x^3y(4+3y)\Big] \\
F^{T}_{2} &= x^3y^2(2+y) \\
F^{Te}_{-1} &= -8\Big[xy(1+3y) + x^2(2+3y) + 2x^3\Big] \\
F^{Te}_{0} &= 4\Big[x^2(2+3y+4y^2) + x^3(2+3y)\Big] \\
F^{Te}_{1} &= -2x^3y(2+y) \\
F^{Te}_{2} &= 0 \\
F^{T\gamma}_{-1} &= -8\Big[y^2(1+y) + xy + x^2y\Big] \\
F^{T\gamma}_{0} &= 4\Big[xy^2(2-y) + x^2y(1+2y) + x^3y\Big] \\
F^{T\gamma}_{1} &= -2\Big[x^2y^2(1-y) + 2x^3y\Big] \\
F^{T\gamma}_{2} &= x^3y^3
\end{aligned}
$$