TWIST (E614) Experiment
Technical Note No.82

Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay

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1 Preamble

This technote deals with the physics of the $\pi^+ \to \mu^+\nu\gamma$ decay. It gives an overview of the problem, a review of the literature, and some results. More detailed calculations and graphs will be presented in another technote.

2 Introduction

The radiative decays of the pion[1]:

$$\pi^\pm \to \mu^\pm\nu\gamma \tag{1}$$

$$\pi^\pm \to e^\pm\nu\gamma \tag{2}$$

and the corresponding kaon decays have been used as a source of information on fundamental interactions and hadronic structure.

The radiative decays of the pion and kaon have been studied by many authors [2, 3, 4, 5, 6, 7, 8, 9].

The observables of these decays (differential rate, polarizations) involve several contributions:

- Internal Bremsstrahlung (IB): the photon is emitted from the electromagnetic current carried by meson and lepton.
- Structure-dependent radiation (SD): the photon is emitted from the intermediate states generated by the strong interaction.

- An interference term (INT) between IB and SD.

The SD and INT contributions are parametrized with form factors. These form factors are calculable in the framework of theoretical models.

The IB term is the product of:

- the non-radiative decay rate $\Gamma(\pi \rightarrow e(\mu)\nu)$, predicted by theory (e.g. the Standard Model) and measured experimentally,
- a term which can be calculated by using Quantum Electro-Dynamics.

This IB contribution does not contain information on hadronic structure. The SD and INT contributions are the ones which contain this kind of information.

The relative importance of the various contributions IB, SD and INT critically depends on the charged-lepton mass. This is because the IB contribution is proportional to the non-radiative rate, which is helicity-suppressed in the Standard Model, in agreement with the experimental result on the $\pi \rightarrow e\nu/\pi \rightarrow \mu\nu$ branching ratio ($\sim 10^{-4}$).

In the electronic mode $\pi \rightarrow e\nu\gamma$, the IB contribution is highly suppressed in a large fraction of the Dalitz plot, and this gives access to valuable information on the structure-dependent contribution[1]. On the other hand, the muon mode $\pi \rightarrow \mu\nu\gamma$ is dominated by Internal Bremsstrahlung. It can be considered, to a good approximation, as a "pure QED process".

The radiative decay of the pion $\pi^+ \rightarrow \mu^+\nu\gamma$, is of interest for the TWIST experiment. One important feature of TWIST is the precise knowledge of the polarization of the surface muons. Inevitably, there is a small contamination of the $\pi^+ \rightarrow \mu^+\nu$ mode by the $\pi^+ \rightarrow \mu^+\nu\gamma$ mode, and this reduces the polarization of the surface muons. The question is: by how much?

According to the Particle Physics Booklet the branching ratio of the radiative muon decay is of the order of 10^{-4}. The most recent experimental value[10] is:

$$BR(E_\gamma > 1 \text{ MeV}) = 1.95 \times 10^{-4} \pm 12.2\% \text{ (stat.)} \pm 4.0\% \text{ (syst.)} \tag{3}$$

to be compared to a theoretical prediction 2.283×10^{-4}. This number is well reproduced by integrating the theoretical probability distribution (see Appendix 2).

Considering the smallness of the branching ratio of the radiative decay, the small fraction of the Dalitz plot which is selected in TWIST, it is possible that we dont have to worry. But we may face some questions.

It should also be said that there is no simple way (maybe no way at all) to detect the photon of the radiative pion decay in the TWIST experiment.

3 Standard Model description

In this technote, we are mainly interested in the IB contribution, for the reason mentioned above. Therefore, the rôle of the other terms SD and INT will be discussed very briefly.

In the Standard Model, these terms contain two form factors, $F_V(q^2)$ and $F_A(q^2)$. These form factors are functions of the momentum transfer squared, but in the low-energy region they are usually taken as constants $F_V = F_V(0)$ and $F_A = F_A(0)$ (for a discussion of the q^2-dependence see [7]). If time-reversal invariance is assumed, the phases of F_V and F_A (with respect to IB) are 0 or π (F_V and F_A are real).

Most of the interesting work concerns the electronic decay mode $\pi^+ \to e^+\nu\gamma$, because of its sensitivity to the SD and INT contributions. In the framework of a model (Standard Model or other), it is possible to compare the measured form factors to theoretical predictions. The vector form factor F_V can be related to the neutral-pion lifetime by using the Conserved-Vector-Current theory (which is an essential ingredient of the Standard Model). The axial-vector form factor F_A can be calculated in a variety of models, some of them inspired by Quantum Chromo-Dynamics.

There is a number of papers which deal with other form factors (for instance a pseudoscalar form factor), but this problem will not be discussed here.

4 The experimental observables

In the following, the muon-neutrino mass is neglected.

Experimentally measurable are:

- The differential decay rate as a function of two kinematic variables. Usually one considers the two invariant variables:

$$x = \frac{2P \cdot q}{M^2} \qquad y = \frac{2P \cdot p}{M^2} \tag{4}$$

  M is the mass of the meson, and the energy-momentum four-vectors are:

  P: Meson (π or K)
  p: Lepton (e or μ)
  q: Photon

  If the meson decays from rest, the variables x and y are the energies of the photon and lepton, expressed in units of half the meson mass:

$$x = \frac{2E_\gamma}{M} \qquad y = \frac{2E_l}{M} \tag{5}$$

  E_γ is the energy of the photon and E_l the total energy of the lepton (including the rest mass). The angle θ between lepton and photon is a function of x and y as we shall see later.

- The polarization vector of the muon. As usual, this vector is defined in the rest frame of the muon, therefore it has only three components:

$$\text{The longitudinal polarization, unit vector } \vec{e}_L = \frac{\vec{p}_\mu}{|\vec{p}_\mu|} \tag{6}$$

$$\text{The normal polarization, unit vector } \vec{e}_N = \frac{\vec{p}_\mu \times (\vec{q} \times \vec{p}_\mu)}{|\vec{p}_\mu \times (\vec{q} \times \vec{p}_\mu)|} \tag{7}$$

$$\text{The transverse polarization, unit vector } \vec{e}_T = \frac{\vec{q} \times \vec{p}_\mu}{|\vec{q} \times \vec{p}_\mu|} \tag{8}$$

The longitudinal component lies along the muon momentum. The other two components are perpendicular to the muon momentum, but:

- The normal component lies in the decay plane (in the pion rest frame, or the laboratory frame if the pion decays at rest),
- The transverse component is perpendicular to the decay plane.

The transverse component can only result from the presence of non-trivial phases between the various form factors. Such phases appear as a consequence of time-reversal violation but they can also be generated by electromagnetic final-state interactions. In any, case the transverse polarization is very small and can be safely neglected for our purposes. Recently, an upper limit has been obtained for the $K^+ \to \mu^+\nu\gamma$ decay[11].

The formulae for the various polarizations contain the same ingredients as the one for the differential rate.

- The polarization vector of the photon. Some theoretical papers have suggested a measurement of the circular polarization of the photon[12]. It is obvious that this would give interesting additional information. Unfortunately, such a measurement does not seem to be feasible.

5 The kinematics

Energy-momentum conservation leads to a relation between x, y and θ:

$$x = \frac{1 - y + r}{1 - \frac{1}{2}y + \frac{1}{2}A\cos\theta} \tag{9}$$

with:

$$r = \frac{m^2}{M^2} \tag{10}$$

where m is the lepton mass, M the meson mass. For the decay of interest:

m	(muon mass)	$= 105.66$ MeV
M	(pion mass)	$= 139.57$ MeV
r		$= 0.573$
$\sqrt{r}$		$= 0.757$

The definition of A is:

$$A = \sqrt{y^2 - 4r} \tag{11}$$

For practical reasons, the above relation can be written as:

$$x = \frac{(1 - \frac{1}{2}y + \frac{1}{2}A)(1 - \frac{1}{2}y - \frac{1}{2}A)}{1 - \frac{1}{2}y + \frac{1}{2}A\cos\theta} \tag{12}$$

because this relation simplifies for $\cos\theta = \pm 1$.

The contour of the kinematically allowed region is given by:

$$y = \frac{(1-x)^2 + r}{1-x} = (1-x) + \frac{r}{(1-x)} \tag{13}$$

and:

$$y = 1 + r \qquad x = \text{ any value} \tag{14}$$

because for $y = 1+r$ and $\cos\theta = -1$ both the numerator and the denominator in the expression of x are zero, therefore x is indeterminate.

Another expression for the equation of the contour is:

$$C(x,y) = (1 - y + r)\Big[(1-x)(x+y-1) - r\Big] = 0 \tag{15}$$

On this contour, the momenta of the three particles are colinear, therefore the decay plane cannot be defined. The particles can only be longitudinally polarized.

Figure 1 shows the kinematics of the radiative $\pi \to \mu\nu\gamma$ decay. The various curves correspond to different values of $\cos\theta$. From left to right:

$\cos\theta$	1.0	0.8	0.6	0.4	0.2	0.0	-0.2	-0.4	-0.6	-0.8	-1.0
θ (degrees)	0	37	53	66	78	90	102	114	127	143	180

Particular points of the contour are:

A	$x = 1 - \sqrt{r}$	$E_\gamma = 16.95$ MeV	$y = 2\sqrt{r}$	$T_\mu = 0.00$ MeV
B	$x = 1 - r$	$E_\gamma = 29.79$ MeV	$y = 1 + r$	$T_\mu = 4.12$ MeV
C	$x = 0.00$	$E_\gamma = 0.00$ MeV	$y = 1 + r$	$T_\mu = 4.12$ MeV

where T_μ is the kinetic energy of the muon.

The singularities observed at points B and C are due to the presence of massless particles. The singularity at point B could be removed by using a finite neutrino mass (which seems to be a physical reality). On the other hand, the singularity at point C cannot be removed. In fact, there is for $x = 0$ a divergence of the decay rate at this point. This divergence is compensated by a similar divergence in the calculation of the radiative correction in the non-radiative decay rate. The presence of this divergence is, of course, a problem for the present work.

The absolute limits for x and y are:

$$0 \leq x \leq 1 - r \tag{16}$$

$$2\sqrt{r} \leq y \leq 1 + r \tag{17}$$

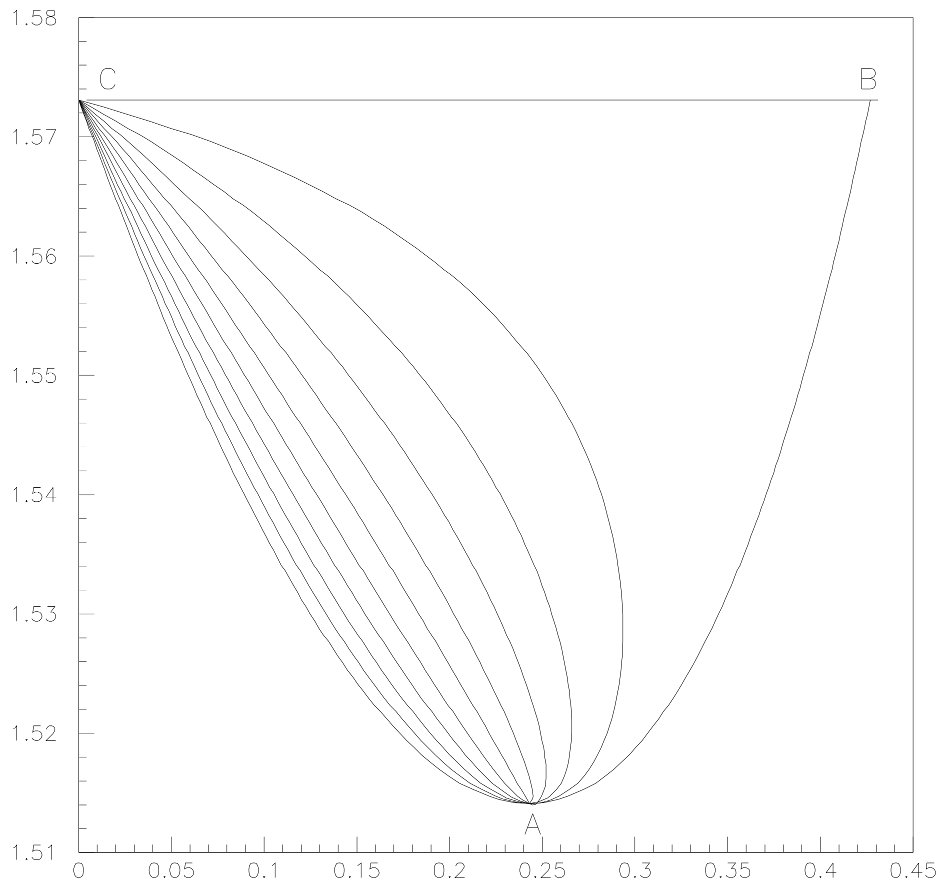

Figure 1: Kinematics of the $\pi \to \mu\nu\gamma$ decay.

But for a given value of x the limits on y are given by:

$$1 - x + \frac{1}{1-x} \leq y \leq 1 + r \tag{18}$$

Another variable λ is used by many authors:

$$\lambda = \frac{x + y - 1 - r}{x} \tag{19}$$

6 The differential decay rate

The relevant papers are those of Bardin and Bilen'kii[6], Bijnens et al.[7] and Chen et al.[9]. In most cases the main interest was in the kaon decays $K \to e(\mu)\nu\gamma$ but the formulae can also be used for the pion decays after minor modifications.

Different notations are used by different authors. For instance:

Bardin gives:

$$\Gamma(\pi \to \mu\nu) = \frac{1}{8\pi} G^2 f_\pi^2 \; m_\pi^3 \; r \; (1-r)^2 \tag{20}$$

Bijnens gives:

$$A_{SD} = \frac{G_F^2 |V_{us}|^2 \alpha}{32\pi^2} m_K^5 = \frac{\alpha}{8\pi} \frac{1}{r(1-r)^2} \left(\frac{m_K}{F_K}\right)^2 \Gamma(K \to l\nu_l) \tag{21}$$

Hence:

$$\Gamma(K \to l\nu_l) = \frac{1}{4\pi} G_F^2 |V_{us}|^2 F_K^2 \; m_K^3 \; r \; (1-r)^2 \tag{22}$$

Bardin gives the rate for the pion decay, therefore the factor $|V_{us}|^2$ must be replaced by $|V_{ud}|^2$ which is close to unity. Since the G used by Bardin is (almost) identical to the G_F used by Bijnens:

$$G = G_F V_{us} \tag{23}$$

one concludes that the f_K used by Bardin is related to the F_K used by Bijnens by:

$$f_k = \sqrt{2} F_K \tag{24}$$

These constants have the dimension of a mass.

Let us now consider the radiative decays. Bardin and Bilen'kii give:

$$\frac{d^2\Gamma}{dxdy} = \left(\frac{\alpha}{32\pi^2}\right) G_F^2 |V_{ud}|^2 m_K^5 \Bigg\{ (2r) \left(\frac{f_K}{m_K}\right)^2 IB(x,y) \tag{25}$$

$$-(2r)\left(\frac{f_K}{m_K}\right)\Big[m_K(a+b)INT^+(x,y) + m_K(a-b)INT^-(x,y)\Big] \tag{26}$$

$$+\frac{1}{2}\Big[m_K^2(a+b)^2 SD^+(x,y) + m_K^2(a-b)^2 SD^-(x,y)\Big]\Bigg\} \tag{27}$$

Let us change notations for the form factors:

$$m_K a = -F_V \tag{28}$$

$$m_K b = -F_A \tag{29}$$

and define a normalization factor:

$$N = \left(\frac{\alpha}{32\pi^2}\right) G_F^2 |V_{us}|^2 m_K^5 \tag{30}$$

The numerical value of N is calculated in Appendix 1.

We now have:

$$\frac{d^2\Gamma}{dxdy} = N\Bigg\{(2r)\left(\frac{f_K}{m_K}\right)^2 IB(x,y) \tag{31}$$

$$+(2r)\left(\frac{f_K}{m_K}\right)\Bigg[(F_V+F_A)INT^+(x,y)+(F_V-F_A)INT^+(x,y)\Bigg] \tag{32}$$

$$+\frac{1}{2}\Bigg[(F_V+F_A)^2 SD^+(x,y)+(F_V-F_A)^2 SD^-(x,y)\Bigg]\Bigg\} \tag{33}$$

with the following expressions for the various functions:

$$IB(x,y) = \frac{1-y+r}{x^2(x+y-1-r)}\left[x^2+2(1-x)(1-r)-\frac{2xr(1-r)}{x+y-1-r}\right] \tag{34}$$

$$INT^+(x,y) = \frac{1-y+r}{x(x+y-1-r)}\Bigg[(1-x)(1-x-y)+r\Bigg] \tag{35}$$

$$INT^-(x,y) = \frac{1-y+r}{x(x+y-1-r)}\Bigg[x^2-(1-x)(1-x-y)-r\Bigg] \tag{36}$$

$$SD^+(x,y) = (x+y-1-r)\Bigg[(x+y-1)(1-x)-r\Bigg] \tag{37}$$

$$SD^-(x,y) = (1-y+r)\Bigg[(1-x)(1-y)+r\Bigg] \tag{38}$$

Taking into account the changes in notation it can be concluded that Bardin and Bilen'kii agree with Bijnens.

If one neglects the lepton mass:

$$IB(x,y) = \frac{(1-y)(x^2-2x+2)}{x^2(x+y-1)} \tag{39}$$

$$INT^+(x,y) = -\frac{(1-x)(1-y)}{x} \tag{40}$$

$$INT^-(x,y) = \frac{1-y}{x}\left[\frac{x^2}{x+y-1}+(1-x)\right] \tag{41}$$

$$SD^+(x,y) = (1-x)(x+y-1)^2 \tag{42}$$

$$SD^-(x,y) = (1-x)(1-y)^2 \tag{43}$$

These expressions have been used for the $\pi \to e\nu\gamma$ decay.

Table 1 shows the population of the Dalitz plot for pure Internal Bremsstrahlung. Horizontal scale: photon energy in MeV. Vertical scale: muon kinetic energy in MeV.

	2.98	5.96	8.94	11.92	14.89	17.87	20.85	23.83	26.82	29.79
4.119	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
3.707	0.1411	0.0171	0.0047	0.0018	0.0009	0.0005	0.0003	0.0002	0.0001	
3.295	0.1910	0.0309	0.0088	0.0034	0.0016	0.0009	0.0005	0.0003	0.0002	
2.884		0.0391	0.0120	0.0048	0.0023	0.0013	0.0008	0.0005	0.0003	
2.472		0.0374	0.0141	0.0059	0.0029	0.0016	0.0010	0.0006	0.0004	
2.060		0.0186	0.0146	0.0067	0.0035	0.0019	0.0011	0.0007		
1.648			0.0126	0.0069	0.0036	0.0021	0.0013	0.0008		
1.237			0.0071	0.0064	0.0037	0.0022	0.0014	0.0009		
0.825				0.0051	0.0035	0.0022	0.0014			
0.413					0.0034	0.0021	0.0014			

Table 1: The differential decay rate (pure IB) as a fonction of x (photon energy in MeV, horizontal scale) and y (muon kinetic energy, vertical scale).

7 The polarization vector of the muon

The longitudinal polarization has been calculated by Bardin and Bilen'kii, and Chen et al., but not by Bijnens et al. Chen et al. have calculated all three components of the polarization vector.

7.1 The longitudinal polarization

7.1.1 Bardin and Bilen'kii

Time-reversal violation is assumed (all form factors are real)

$$\vec{P} = P\left(\frac{\vec{v}}{v}\right) \tag{44}$$

where $\vec{v}$ is the velocity of the muon and P the degree of longitudinal polarization.

$$P = -\left(\frac{\alpha}{32\pi^2}\right)G_F^2|V_{ud}|^2 m_K^5\left(\frac{d^2\Gamma}{dxdy}\right)^{-1}\frac{\sqrt{y^2-4r}}{y}\Bigg\{(2r)\left(\frac{f_K}{m_K}\right)^2 PIB(x,y) \tag{45}$$

$$-(2r)\left(\frac{f_K}{m_K}\right)\Bigg[m_K(a+b)PINT^+(x,y)+m_k(a-b)PINT^-(x,y)\Bigg] \tag{46}$$

$$+\frac{1}{2}\Bigg[m_K^2(a+b)^2PSD^+(x,y)+m_K^2(a-b)^2PSD^-(x,y)\Bigg]\Bigg\} \tag{47}$$

where the quantity

$$\frac{d^2\Gamma}{dxdy} \tag{48}$$

is the differential decay rate.

After a change of notations for the form factors:

$$P=-N\left(\frac{d^2\Gamma}{dxdy}\right)^{-1}\frac{\sqrt{y^2-4r}}{y}\Bigg\{(2r)\left(\frac{f_K}{m_K}\right)^2 PIB(x,y) \tag{49}$$

$$+(2r)\left(\frac{f_K}{m_K}\right)\Bigg[(F_V+F_A)PINT^+(x,y)+(F_V-F_A)PINT^-(x,y)\Bigg] \tag{50}$$

$$+\frac{1}{2}\Bigg[(F_V+F_A)^2PSD^+(x,y)+(F_V-F_A)^2PSD^-(x,y)\Bigg]\Bigg\} \tag{51}$$

with the following expressions for the various functions:

$$PIB(x,y)=\frac{1-y+r}{x^2(x+y-1-r)}\Bigg[x^2+2(1+r-x)-\frac{2xyr}{x+y-1-r} \tag{52}$$

$$+2r\frac{xy-2(x+y-1-r)}{y^2-4r}\left(2-x-\frac{(1+r)x}{x+y-1-r}\right)\Bigg] \tag{53}$$

$$PINT^+(x,y)=\frac{1-y+r}{x(x+y-1-r)}\Bigg[(1-x)(x+y-1)-r\Bigg]\frac{y(2-y-2x)}{y^2-4r} \tag{54}$$

$$PINT^-(x,y)=\frac{1-y+r}{x(x+y-1-r)} \tag{55}$$

$$\Bigg[-(1-x)(1-y)-r+\frac{xy-2(x+y-1-r)}{y+2r}\Bigg] \tag{56}$$

$$PSD^+(x,y)=\Bigg[(x+y-1-r)(x-1)+rx\Bigg] \tag{57}$$

$$\Bigg[x+y-1-r-2r\ \frac{xy-2(x+y-1-r)}{y^2-4r}\Bigg] \tag{58}$$

$$PSD^-(x,y)=(1-y+r)\Bigg[-(1-x)(1-y)-r-2r\ \frac{xy-2(x+y-1-r)}{y^2-4r}\Bigg] \tag{59}$$

Table 2 shows the longitudinal polarization (pure Internal Bremsstrhlung) as a function of the photon energy (horizontal scale, in MeV) and the muon kinetic energy (vertical scale, in MeV).

It could be concluded, from the table, that the longitudinal polarization takes negative values everywhere in the Dalitz plot. This is not exactly the case but it is not easy to present a table or a graph which clearly shows the region where the longitudinal polarization takes positive values. This point is discussed in Appendix 3: there is, in fact, a region of the Dalitz plot where the longitudinal polarization takes positive values, including $P_L=+1$. But this is a very narrow part of the Dalitz plot.

	2.98	5.96	8.94	11.92	14.89	17.87	20.85	23.83	26.82	29.79
4.119	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000	-1.000
3.707	-0.998	-0.997	-0.995	-0.994	-0.993	-0.992	-0.993	-0.995	-0.997	
3.295	-0.992	-0.993	-0.990	-0.987	-0.985	-0.984	-0.986	-0.989	-0.995	
2.884		-0.985	-0.983	-0.979	-0.976	-0.976	-0.979	-0.985	-0.995	
2.472		-0.958	-0.971	-0.969	-0.966	-0.967	-0.971	-0.982	-0.995	
2.060		-0.742	-0.944	-0.952	-0.953	-0.955	-0.964	-0.980	-1.000	
1.648			-0.856	-0.918	-0.931	-0.941	-0.956	-0.968		
1.237			-0.384	-0.833	-0.891	-0.919	-0.949	-0.990		
0.825				-0.546	-0.796	-0.880	-0.945			
0.413					-0.497	-0.802	-0.971			

Table 2: The longitudinal polarization of the muon (pure IB) as a function of x (photon energy in MeV, horizontal scale) and y (muon kinetic energy in MeV, vertical scale).

7.1.2 Chen, Geng and Lih

The three components of the polarization vector are given as ratios:

$$P_i(x,y) = \frac{\rho_i(x,y)}{\rho_0(x,y)} \qquad (i = L, N, T) \tag{60}$$

where L, N and T refer to the longitudinal, normal and transverse components, respectively.

$$\begin{aligned}
\rho_0(x,y) = {} & \frac{1}{2}e^2 G_F^2 \sin^2\theta_C (1-\lambda)\left\{ \frac{4m_\mu^2 |f_K|^2}{\lambda x^2}\rho_{0IB}(x,\lambda) \right. \\
& + m_\pi^4 x^2 |F_V + F_A|^2 \rho_{0SD^+}(x,\lambda) \\
& + m_\pi^4 x^2 |F_V - F_A|^2 \rho_{LSD^-}(x,\lambda) \\
& - 4 m_\pi m_\mu^2 \mathcal{R}e[F_\pi (F_V + F_A)^*] \rho_{0INT^+}(x,\lambda) \\
& \left. + 4 m_\pi m_\mu^2 \mathcal{R}e[F_\pi (F_V - F_A)^*] \rho_{0INT^-}(x,\lambda) \right\}
\end{aligned} \tag{61}$$

with:

$$\rho_{0IB}(x,\lambda) = x^2 + 2(1-r)\left(1 - x - \frac{r}{\lambda}\right) \tag{62}$$

$$\rho_{0SD^+}(x,\lambda) = \frac{\lambda^2}{1-\lambda}\left(1 - x - \frac{r}{\lambda}\right) \tag{63}$$

$$\rho_{0SD^-}(x,\lambda) = (y - \lambda) \tag{64}$$

$$\rho_{0INT^+}(x,\lambda) = \left(1 - x - \frac{r}{\lambda}\right) \tag{65}$$

$$\rho_{0INT^-}(x,\lambda) = \frac{1 - y + \lambda}{\lambda} \tag{66}$$

$$
\begin{aligned}
\rho_L(x,y) = -\frac{1}{2}e^2 G_F^2 \sin^2\theta \frac{(1-\lambda)}{\lambda\sqrt{y^2-4r}} \Bigg\{ & \frac{4m_\mu^2|f_K|^2}{\lambda x^2}\rho_{LIB}(x,\lambda) \\
& -m_\pi^4 \lambda x^2 |F_V + F_A|^2 \rho_{LSD^+}(x,\lambda) \\
& -m_\pi^4 \lambda x^2 |F_V - F_A|^2 \rho_{LSD^-}(x,\lambda) \\
& -4m_\pi m_\mu^2 \mathcal{R}e[F_\pi(F_V+F_A)^*]\rho_{LINT^+}(x,\lambda) \\
& -4m_\pi m_\mu^2 \mathcal{R}e[F_\pi(F_V-F_A)^*]\rho_{LINT^-}(x,\lambda) \Bigg\}
\end{aligned} \tag{67}
$$

with:

$$\rho_{LIB}(x,\lambda) = x(\lambda y - 2r)(x+y-2\lambda) - (y^2-4r)(\lambda x + 2r - 2\lambda) \tag{68}$$

$$\rho_{LSD^+}(x,\lambda) = \frac{\lambda}{1-\lambda}(\lambda y - 2r)\left(1 - x - \frac{r}{\lambda}\right) \tag{69}$$

$$\rho_{LSD^-}(x,\lambda) = (y^2 - \lambda y - 2r) \tag{70}$$

$$\rho_{LINT^+}(x,\lambda) = \lambda\left(1 - x - \frac{r}{\lambda}\right)(2 - 2x - y) \tag{71}$$

$$\rho_{LINT^-}(x,\lambda) = (1-y)(y-\lambda) + 2r - \lambda \tag{72}$$

The definition of λ has been given in equation 19.

7.2 The full polarization vector

Chen, Geng and Lih[9] have calculated the three components of the polarization vector for the decay $K^+ \rightarrow \mu^+\nu\gamma$. Their formulae can also be used for the $\pi^+ \rightarrow \mu^+\nu\gamma$ with minor modifications. Their primary interest was in the transverse polarization (perpendicular to the decay plane). This component is time-reversal violating but also subject to final-state effects. An upper limit has been obtained recently[11]. For the purpose of the TWIST experiment this transverse component can be safely neglected.

The formulae given by Chen et al. look different from those of Bardin and Bilen'kii because of different notations but they should give identical results. This is the case for the longitudinal polarization in the case of pure IB. More complete checks are in progress.

8 The program MUPOL

This program calculates the differential decay rate and longitudinal polarization P_L as a function of x and y, using the formulae of Bardin et al.[6]. This program is available from the author of this technote.

The formulae given by Chen et al.[9] will also be used, later, for the normal polarization vector P_N.

To avoid problems in the numerical calculation, some particular cases must be treated separately. Some expressions are singular for:

$$x = 1 - \sqrt{r} \qquad y = 2\sqrt{r} \tag{73}$$

It is possible to show that, in fact, they have finite values at this point. This can be shown by studying their limits for:

$$y = 2\sqrt{r} + \epsilon \qquad \epsilon \rightarrow 0 \tag{74}$$

For instance:

$$g = \frac{xy - 2(x + y - 1 - r)}{y^2 - 4r} \rightarrow -\frac{1 + \sqrt{r}}{4\sqrt{r}} \tag{75}$$

$$h = \frac{y(2 - 2x - y)}{y^2 - 4r} \rightarrow -\frac{1}{2} \tag{76}$$

All terms $PIB(x, y)$, $PINT^+(x, y)$, $PINT^-(x, y)$, $PSD^+(x, y)$ and $PSD^-(x, y)$ are finite, except for $x = 0$.

9 Conclusion

Although it is not quite clear how to handle the problem of the divergence for $x = 0$, a simple analysis shows that the effect of the radiative pion decay on the muon polarization used in TWIST is, most likely, substantially smaller than 10^{-4} and should not be a great concern for us. Further discussion in the collaboration would be desirable.

The present numerical calculations are for a pure IB process, and this should be close to reality. Nevertheless, the SD and INT contributions will also be investigated and the results presented in another technote.

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10 Appendix 1: the normalization factor N

For the calculation of N (defined in the section on the differential rate):

$$N = \left(\frac{\alpha}{32\pi^2}\right) G_F^2 |V_{ud}|^2 m_\pi^5 \tag{77}$$

let us use the following data:

$$\alpha = 1/137.036 \tag{78}$$

$$G_F = 1.1664 \times 10^{-5}\ \mathrm{GeV}^{-2} \tag{79}$$

$$m_\pi = 0.13957\ \mathrm{GeV} \tag{80}$$

$$V_{ud} = 0.975 \tag{81}$$

The result is:

$$N = 1.583 \times 10^{-19}\ \mathrm{GeV} \tag{82}$$

using $\hbar c = 0.1973$ GeV-fm:

$$N = 8.021 \times 10^{-19}\ \mathrm{fm}^{-1} \tag{83}$$

and $c = 2.998 \times 10^{23}$ fm-s^{-1}:

$$N = 2.405 \times 10^{5}\ \mathrm{s}^{-1} \tag{84}$$

11 Appendix 2: Integration of Dalitz plot

A numerical integration has been performed for different sections of the Dalitz plot. Since there is a divergence for x (photon energy) $= 0$, a cut-off has been introduced. In order to compare to the value quoted by Bressi et al.[10] a cut-off in photon energy of 1.00 MeV has been chosen. The branching ratio quoted by these authors (2.283×10^{-4}) is well reproduced (2.282×10^{-4}).

The upper bound of integration is kept as 1.5731 (in units of m_π), corresponding to a kinetic energy of 4.12 MeV. The lower bound is varied by steps.

Limits of integration for photon	xmin = 0.01433	1.00 MeV	
	xmax = 0.42689	29.79 MeV	
Limits of integration for muon	ymin = 1.51408	105.66 MeV	Tmin = 0.00 MeV
	ymax = 1.57311	109.78 MeV	Tmax = 4.12 MeV

1.5141	0.0000	100%	2.282×10^{-4}
1.5259	0.8249	80%	2.244×10^{-4}
1.5377	1.6484	60%	2.142×10^{-4}
1.5495	2.4719	40%	1.923×10^{-4}
1.5613	3.2953	20%	1.390×10^{-4}
1.5672	3.7071	10%	0.724×10^{-4}
1.5702	3.9164	5%	0.210×10^{-4}
1.5725	4.0769	1%	0.010×10^{-4}
1.5731	4.1194	0%	0.000×10^{-4}

1st column: lower bound of integration in units of $m_\pi/2$.
2nd column: lower bound of integration in MeV.
3rd column: fractional energy range for muon.
4th column: integrated branching ratio.

12 Appendix 3: P_L distribution

The following tables can be used to understand the distribution of the longitudinal polarization of the muon in the Dalitz plot.

Table 3 shows the values of x (photon energy, column 1) and y (muon kinetic energy, columns 2, 3 and 4), in units of $m_\pi/2$ for:

- muon longitudinal polarization equal to -0.9 (column 2),
- muon longitudinal polarization equal to 0.0 (column 3),
- lowest muon energy allowed by kinematics (column 4).

In order to see the very small differences between columns 3 and 4 it was desirable to print many more digits than necessary.

Negative values for the longitudinal muon polarization P_L occupy the largest part of the Dalitz plot (the maximum value for the muon kinetic energy y is 1.5731). It should also be noted that there are no positive values of P_L for photon energies x larger than $1 - \sqrt{r} = 0.243$.

Table 4 shows the values of x (photon energy, column 1) and y (muon kinetic energy, columns 2, 3) in units of $m_\pi/2$ for:

- muon longitudinal polarization equal to 0.0 (column 2),
- muon longitudinal polarization equal to 1.0 (column 3),

Column 4 contains the difference between columns 2 and 3. This shows that positive longitudinal polarizations are restricted to a very narrow region of the Dalitz plot. In order to see the very small differences between columns 2 and 3 it was desirable to print many more digits than necessary.

x	y for $PL = -0.9$	y for $PL = 0.0$	$y_{min}(x)$
0.01	1.5689106	1.5688984	1.5688977
0.02	1.5649068	1.5648104	1.5648049
0.03	1.5611668	1.5608528	1.5608338
0.04	1.5577374	1.5570339	1.5569883
0.05	1.5546382	1.5533622	1.5532724
0.06	1.5518653	1.5498458	1.5496902
0.07	1.5493997	1.5464926	1.5462460
0.08	1.5472146	1.5433098	1.5429443
0.09	1.5452808	1.5403031	1.5397899
0.10	1.5435703	1.5374769	1.5367875
0.11	1.5420566	1.5348337	1.5339424
0.12	1.5407159	1.5323736	1.5312600
0.13	1.5395262	1.5300946	1.5287457
0.14	1.5384673	1.5279920	1.5264055
0.15	1.5375198	1.5260587	1.5242456
0.16	1.5366650	1.5242856	1.5222723
0.17	1.5358840	1.5226617	1.5204925
0.18	1.5351571	1.5211746	1.5189131
0.19	1.5344633	1.5198110	1.5175417
0.20	1.5337794	1.5185568	1.5163860
0.21	1.5330792	1.5173979	1.5154541
0.22	1.5323319	1.5163202	1.5147548
0.23	1.5314991	1.5153098	1.5142971
0.24	1.5305298	1.5143536	1.5140905
0.25	1.5293489	———	1.5141450
0.26	1.5278258	———	1.5144713
0.27	1.5256554	———	1.5150805
0.27	1.5158198	———	1.5150805
0.26	1.5146485	———	1.5144713
0.25	1.5141651	———	1.5141450

Table 3: Values of x (photon energy) and y (muon kinetic energy) for muon longitudinal polarizations -0.9 and 0.0. The last column shows the minimum value of y allowed by kinematics.

x	y for $P_L = 1$	y for $P_L = 0$	Δy
0.01	1.568898	1.568898	0.000000
0.02	1.564805	1.564810	0.000005
0.03	1.560834	1.560853	0.000019
0.04	1.556988	1.557034	0.000046
0.05	1.553272	1.553362	0.000090
0.06	1.549690	1.549846	0.000156
0.07	1.546246	1.546493	0.000247
0.08	1.542944	1.543310	0.000366
0.09	1.539790	1.540303	0.000513
0.10	1.536788	1.537477	0.000689
0.11	1.533942	1.534834	0.000892
0.12	1.531260	1.532374	0.001114
0.13	1.528746	1.530095	0.001349
0.14	1.526406	1.527992	0.001586
0.15	1.524246	1.526059	0.001813
0.16	1.522272	1.524286	0.002014
0.17	1.520493	1.522662	0.002169
0.18	1.518913	1.521175	0.002262
0.19	1.517542	1.519811	0.002269
0.20	1.516386	1.518557	0.002171
0.21	1.515454	1.517398	0.001944
0.22	1.514755	1.516320	0.001565
0.23	1.514297	1.515310	0.001013
0.24	1.514091	1.514354	0.000263

Table 4: Photon energies (x, column 1) and muon kinetic energies (y) for $P_L = 1$ (column 2) and $P_L = 0$ (column 3) and their differences (Δy, column 4).