Systematic errors in statistical measurements.

$\mathcal{TWIST}$ Technical note #94

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Abstract

This note addresses the current debate at $\mathcal{TWIST}$ about the correct way to calculate the systematic error. The approach used here is to calculate the total error, then identify the statistical error term in the expression. What is left is the systematic error.

Let us consider an experiment designed to measure some quantity R_{true}. From the experiment we get a value R which is an estimator of R_{true}. Suppose the experiment is done correctly and R is an unbiased estimator of R_{true}:

$$\mathrm{E}[R] = R_{\text{true}}, \tag{1}$$

where E[] is the expectation value operator. Then the *total* error can be defined as the square root of the variance

$$\sigma_{\text{R,total}} = \sqrt{\mathrm{V}[R]}, \tag{2}$$

with V[] being the variance operator.

Systematic errors arise when result depends on imprecisely known parameters. (E.g. calibrations.) I.e. $R = R(x)$ is a random function of random variable x. Both R and x in this note are scalars. That is, we are measuring one number, and there is one systematics (related to x) affecting it. The generalization for the case of several systematics is obvious.

We will be using the standard approximation of linearity of R in x on the relevant interval [1]:

$$R(x) = R(x_0) + R'(x_0)\,(x - x_0). \tag{3}$$

Let us calculate the total error on R. If $R(x)$ is a deterministic function (e.g. given by a formula) $R(x_0)$ and $R'(x_0)$ are constants. Then the usual error propagation rule [1] can be derived:

$$\begin{aligned}
(4)\qquad \mathrm{V}[R] &= \mathrm{V}[R(x_0) + R'(x_0)\,(x - x_0)] && \text{By (3)}\\
(5)\qquad &= \mathrm{V}[R(x_0)] + \mathrm{V}[R'(x_0)\,(x - x_0)] && \text{By (28)}\\
(6)\qquad &= 0 + {R'}^2(x_0)\,\mathrm{V}[x - x_0] && \text{By (25),(26)}\\
(7)\qquad &= {R'}^2(x_0)(\mathrm{V}[x] - 0) && \text{By (28),(25)}\\
(8)\qquad &= {R'}^2(x_0)\ \mathrm{V}[x]
\end{aligned}$$

In the other limiting case $R(x)$ is a random function, but x is precisely known: $\mathrm{V}[x] = 0$, $x = x_0$, $\mathrm{V}[R(x)] = \mathrm{V}[R(x_0)]$. In this case systematic error is absent, and we can identify statistical error as

$$(9)\qquad \sigma_{\mathrm{R,stat}} = \sqrt{\mathrm{V}[R(x_0)]}$$

Now consider the general case when both statistical and systematic terms are important. Here we do not assume $\mathrm{V}[R(x_0)] = 0$, in the same way $\mathrm{V}[R'(x_0)] \neq 0$.

$$(10)\qquad \mathrm{V}[R] = \mathrm{V}[R(x_0) + R'(x_0)\,(x - x_0)]$$

$R(x_0)$ and x are always independent, since their fluctuations come from different sources (event statistics vs e.g. high voltage). Assuming that $R(x_0)$ and $R'(x_0)$ are independent,

$$(11)\qquad = \mathrm{V}[R(x_0)] + \mathrm{V}[R'(x_0)\,(x - x_0)]$$

using the definition of variance (21)

$$(12)\qquad = \mathrm{V}[R(x_0)] + \mathrm{E}[(R'(x_0)\,(x - x_0))^2] - \mathrm{E}^2[R'(x_0)\,(x - x_0)]$$

$R'(x_0)$ and x are independent, use (27)

$$\begin{aligned}
(13)\qquad &= \mathrm{V}[R(x_0)] + \mathrm{E}[{R'}^2(x_0)]\ \mathrm{E}[(x - x_0)^2] - \mathrm{E}^2[R'(x_0)]\ \mathrm{E}^2[(x - x_0)]\\
(14)\qquad &= \mathrm{V}[R(x_0)] + \mathrm{E}[{R'}^2(x_0)]\ \mathrm{V}[x] - \mathrm{E}^2[R'(x_0)]\,0
\end{aligned}$$

express $\mathrm{E}[{R'}^2(x_0)]$ through $\mathrm{V}[R'(x_0)]$ and $\mathrm{E}[R'(x_0)]$ using (21)

$$\begin{aligned}
(15)\qquad &= \mathrm{V}[R(x_0)] + \big(\mathrm{E}^2[R'(x_0)] + \mathrm{V}[R'(x_0)]\big)\ \mathrm{V}[x] - 0\\
(16)\qquad &= \mathrm{V}[R(x_0)] + \mathrm{E}^2[R'(x_0)]\ \mathrm{V}[x] + \mathrm{V}[R'(x_0)]\ \mathrm{V}[x]
\end{aligned}$$

The first term in (16) is the statistical error, the second is the usual "error propagation" part of the systematics. This second term can only be reduced by a better control of experimental variable x. But there is also another contribution to the systematic error, the product of a "hardware systematic" $\mathrm{V}[x]$ (the uncertainty in e.g. the high voltage) and a "statistical error on the derivative"

$V[R'(x_0)]$. That term can be reduced by either better controlling x, or by better measuring $R'(x_0)$ by e.g. using more MC (or systematic data set) statistics.

In $\mathcal{TWIST}$ we get an estimate of a Michel parameter $E[R(x_0)]$ as the central value of the "data to MC" fit, with the fit error being an estimate of $V[R(x_0)]$. The central value of a systematic fit with an exaggeration factor $S = \Delta x_{\text{test}}/\sqrt{V[x]}$ gives us an estimate of $\Delta x_{\text{test}}\, E[R'(x_0)]$ (the systematic bias of the result), while the error on the bias is an estimate of $(\Delta x_{\text{test}})^2\ V[R'(x_0)]$.

We can rewrite (16) as

$$\sigma^2_{\text{R,total}} = \sigma^2_{\text{R,stat}} + \sigma^2_{\text{R,sys}} \tag{17}$$

where

$$\sigma^2_{\text{R,stat}} = V[R(x_0)] \tag{18}$$

and

$$\sigma^2_{\text{R,sys}} = E^2[R'(x_0)]\, V[x] + V[R'(x_0)]\, V[x] \tag{19}$$

$$= \frac{1}{S^2}(\text{bias}^2 + \text{error}^2) \tag{20}$$

I.e. the systematic error is the scaled quadratic sum of the central value and the error of the corresponding systematic fit.

Appendix

The definition of variance:

$$V[a] = E[a^2] - E^2[a] \tag{21}$$

where $E[R]$ is the expectation value.

For any variables a, b, and a constant C, the following relations are true:

$$E[C] = C \tag{22}$$

$$E[Ca] = C\, E[a] \tag{23}$$

$$E[a+b] = E[a] + E[b] \tag{24}$$

$$V[C] = 0 \tag{25}$$

$$V[Ca] = C^2\, V[a] \tag{26}$$

If a and b are independent,

$$E[ab] = E[a]\, E[b] \tag{27}$$

$$V[a+b] = V[a] + V[b] \tag{28}$$

References

[1] W.T.Eadie, D.Drijard, F.E.James, M.Roos, B.Sadoulet, *"Statistical methods in experimental physics"*, North-Holland, 1971.