TWIST (E614) Experiment
Technical Note 96

Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay
Supplement to Technical Notes 82 and 89

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April 16th, 2005

1 Preamble

This technote is a supplement to Technotes 82 [1] and 89 [2]. Rather than updating these two technotes, it is more practical to present the additional information in a separate technote.

This new technote contains a lot of details and the busy reader may just like to have a look at the two tables, at the end of the paper.

What is new in the present technote? The contributions of the Structure-Dependent ($SD^{\pm}$) terms and their interference (IB-$SD^{\pm}$) with the Internal Bremstrahlung (IB) contribution are now taken into account. As expected, the contribution of these additional terms is negligible, therefore the conclusion of Technotes 82 and 89 is still valid.

2 What is the problem?

The question addressed in Technotes 82 and 89 was: how would the radiative muon decay $\pi^+ \to \mu^+\nu\gamma$ affect muon polarization, which, according to the Standard Model, is $P_L = -1$ for the normal decay $\pi^+ \to \mu^+\nu$? (the positive-muon polarization is longitudinal and reaches its maximum negative value -1). It is reasonable to expect that the contamination of the normal muon decay by the radiative decay would change the muon polarization. But, how important would the effect be for TWIST? In the two previous technotes it was concluded that this effect is very small and can be neglected. The conclusion was based on the following arguments:

- The radiative muon decay has a small branching ratio. A recent measurement [3] gives:

$$\text{B.R.}(\pi^+ \to \mu^+\nu\gamma, E_\gamma > 1\text{MeV}) = 2.0 \times 10^{-4} \pm 0.24(\text{stat.}) \pm 0.08(\text{syst.})$$

  The corresponding theoretical B.R. is 2.283×10^{-4}.

- The TWIST experiment uses a small part of the Dalitz plot (a few per-cent).

- The longitudinal muon polarization is close to -1 in a large fraction of the Dalitz plot, especially in that part of the Dalitz plot which is used in the experiment.

This conclusion results from the fact that the Internal Bremstrahlung (IB) contribution is the dominant one in the muonic mode (it is not true for the electronic mode). Therefore, it is possible to neglect the Structure-Dependent effects ($SD^\pm$), as well as the interference terms (IB-$SD^\pm$). This was the basis of the calculations done in technotes 82 and 89.

However, the following question may arise: if the $SD^\pm$ and IB-$SD^\pm$ terms are taken into account, is it possible for the muon polarization to take the value $+1$ in a region of the Dalitz plot which contributes to TWIST?

The answer is "yes" but this does not change the previous conclusion: it is a negligible effect. This happens when the muon energy (in the radiative decay) reaches its maximal value (the non-radiative value). There, the IB, IB-$SD^\pm$ and SD^- contributions vanish, but not the SD^+ contribution which produces a muon polarization $P = +1$. This is the only situation where the SD^+ contribution can compete whith the other contributions, but, as we shall see, this SD^+ contribution produces a negligible effect.

In the following, I take the opportunity to give a detailed description of the physics of radiative meson decays (which includes pion and kaon decays into electron and muon).

3 The branching ratio of the radiative muon decay

The value which has been given in the litterature for a long time [4]:

$$\text{B.R.} = (1.24 \pm 0.25) \times 10^{-4}$$

is based on the emulsion experiment of Castagnoli and Muchnik [5]. This branching ratio corresponds to a cut of $< 435\mu$ on the muon track (a cut on the muon kinetic energy $T_\mu < 3.38$ MeV).

The most recent experimental value [3] is:

$$\text{B.R.}(E_\gamma > 1 \text{ MeV}) = 2.0 \times 10^{-4} \pm 0.24 \text{ (stat.)} \pm 0.08 \text{ (syst.)} \tag{1}$$

to be compared with a theoretical prediction 2.283×10^{-4}, assuming a cut-off $E_\gamma < 1$ MeV.

This number is well reproduced by integrating the theoretical probability distribution (see Appendix 1 of Technote 82) based on the Internal Bremsstrahlung contribution (IB) only.

The value obtained by Castagnoli and Muchnik is also consistent with the theoretical prediction, assuming the above energy cut-off $T_\mu < 3.38$ MeV.

In contrast to the electronic decay mode $\pi^+ \to e^+\nu\gamma$, the muonic decay mode $\pi^+ \to \mu^+\nu\gamma$ is not helicity suppressed. It is dominated by the Internal Bremsstrahlung (IB) contribution. The Structure-Dependent ($SD^\pm$) contributions can be neglected (also the interference terms (IB-$SD^\pm$) between ID and $SD^\pm$ can be neglected). The experimental result of Bressi et al. [3] is in agreement with this conclusion, not only in the branching ratio but also in the Dalitz plot distribution.

4 The kinematics

Let us recall the most important features of the kinematics.

An important parameter is the ratio:

$$r = \frac{m^2}{M^2}$$

where m is the lepton mass, M the meson mass. For the decay of interest:

$$\begin{array}{lll} m & \text{(muon mass)} & = 105.66 \text{ MeV} \\ M & \text{(pion mass)} & = 139.57 \text{ MeV} \\ r & & = 0.573 \\ \sqrt{r} & & = 0.757 \end{array}$$

Be careful! In the paper of Bressi et al., the parameter r has a different definition:

$$r = \frac{m}{M}$$

The contour of the kinematically allowed region is given by:

$$y = (1-x) + \frac{r}{(1-x)}$$

and:

$$y = 1 + r \qquad x = \text{ any value}$$

or:

$$C(x,y) = (1-y+r)\Big[(1-x)(x+y-1) - r\Big] = 0$$

On this contour, the momenta of the three particles are colinear, therefore the decay plane cannot be defined. The muon must be longitudinally polarized.

More details on the kinematics can be found in Technote 82 [1].

5 The differential decay rate

The relevant papers are those of Bardin and Bilen'kii [6], Bijnens et al. [7], Bijnens et al. [8], Bijnens et al. [9], and Chen et al. [10]. See also the Particle Data Group booklet [11]. In most cases the interest was in the kaon decays $K \to e(\mu)\nu\gamma$, but the formulae can also be used for the pion decays $\pi \to e(\mu)\nu\gamma$ after minor modifications: the ratio r has to be changed and the CKM matrix element V_{ud} must be used instead of V_{us}.

Different notations are used by different authors. For instance, Chen et al. use the variable:

$$\lambda = \frac{x + y - 1 - r}{x}$$

In the following the formulae of Bardin and Bilen'kii [6] will be used.
The rate for the normal decay $\pi^+ \to \mu^+\nu$ is given by:

$$\Gamma = \frac{1}{8\,\pi}\,G^2\,f_\pi^2\,m_\pi^3\,r\,(1-r)^2 = \frac{1}{8\,\pi}\,G_F^2\,|V_{ud}|^2\,f_\pi^2\,m_\pi^3\,r\,(1-r)^2 \qquad \text{with } r = \left(\frac{m_\mu}{m_\pi}\right)^2$$

(assuming that the factor $|V_{ud}|^2$ has been taken into account in the definition of G). There is another source of information: Review of Particle Physics, Phys. Rev. D, 66, 010001-439.

$$\begin{aligned}\Gamma &= \frac{1}{8\,\pi}\,G_F^2\,|V_{ud}|^2\,f_\pi^2\,m_\mu^2\,m_\pi\left(1-\frac{m_\mu^2}{m_\pi^2}\right)^2\Big[1+\mathcal{O}(\alpha)\Big]\\ &= \frac{1}{8\,\pi}\,G_F^2\,|V_{ud}|^2\,f_\pi^2\,m_\pi^3\,r\,(1-r)^2\,\Big[1+\mathcal{O}(\alpha)\Big]\end{aligned}$$

It is in order to do a numerical check, using the values of reference [12]:

G_F	1.16639×10^{-5}	GeV^{-2}	page	010001-77
V_{ud}	0.9748		page	010001-113
m_μ	0.105658	GeV	page	010001-350
m_π	0.139570	GeV	page	010001-440
f_π	0.1307	GeV	page	010001-439
$\hbar$	6.582×10^{-25}	GeV-s	page	010001-77

The result is:

$$\tau_\pi = \frac{1}{\Gamma} = 2.638 \times 10^{-8}\ \text{s.}$$

close to the measured value:

$$\tau_\pi = \frac{1}{\Gamma} = 2.603 \times 10^{-8}\ \text{s.}$$

The difference can be explained by the radiative correction $\mathcal{O}(\alpha)$.

Let us now consider the radiative decays. Using the formulae of Bardin and Bilen'kii for the π meson:

$$\frac{d^2W}{dxdy} = \left(\frac{\alpha}{2\ \pi}\right)\left(\frac{1}{8\ \pi}\ G^2\ m_\pi\ m_\mu^2\ f_\pi^2\right)\Bigg\{IB(x,y)$$

$$+\left(\frac{m_\pi}{f_\pi}\right)(m_\pi\ a)\left[\left(1+\frac{b}{a}\right)IB\text{-}SD^+(x,y)-\left(1-\frac{b}{a}\right)IB\text{-}SD^-(x,y)\right]$$

$$+\frac{1}{4}\left(\frac{m_\pi}{m_\mu}\right)^2\left(\frac{m_\pi}{f_\pi}\right)^2(m_\pi\ a)^2\left[\left(1+\frac{b}{a}\right)^2SD^+(x,y)+\left(1-\frac{b}{a}\right)^2SD^-(x,y)\right]\Bigg\}$$

where $m_\pi\ a$ is dimensionless.

We have the following expressions for the various kinematical functions:

$$IB(x,y) = \frac{1-y+r}{x^2(x+y-1-r)}\left[x^2+2(1-x)(1-r)-\frac{2xr(1-r)}{x+y-1-r}\right]$$

$$IB\text{-}SD^+(x,y) = -\frac{1-y+r}{x(x+y-1-r)}\Big[(1-x)(1-x-y)+r\Big]$$

$$IB\text{-}SD^-(x,y) = -\frac{1-y+r}{x(x+y-1-r)}\Big[x^2-(1-x)(1-x-y)-r\Big]$$

$$SD^+(x,y) = (x+y-1-r)\Big[(x+y-1)(1-x)-r\Big]$$

$$SD^-(x,y) = (1-y+r)\Big[(1-x)(1-y)+r\Big]$$

Note that the contributions $IB, IB\text{-}SD^+, IB\text{-}SD^-$ and SD^- vanish for $y = 1+r$, when the energy of the muon reaches its maximum (this corresponds to the energy of the muon in the non-radiative decay).

For $y = 1+r$

$$SD^+ = x^2(1-x-r)$$

This vanishes for $x = 0$ and $x = 1+r$ and reaches a maximum for

$$x = \frac{2(1-r)}{3} = 0.2846$$

$$(SD^+)_{max} = \frac{4}{27}(1-r)^3 = 0.0115$$

The double-differential decay rate can also be written as:

$$\frac{d^2W}{dxdy} = N(2r)\left(\frac{f_\pi}{m_\pi}\right)^2\Big\{IB(x,y)+...\Big\}$$

with:

$$N = \left(\frac{\alpha}{32\pi^2}\right) G_F^2 |V_{ud}|^2 m_\pi^5$$

The numerical value of N is calculated in Appendix 1 of [1].

Two important remarks:

- There is a singularity for $x = 0$ (x is the photon energy). This is characteristic of radiative processes. This singularity is compensated by a similar singularity which arises in the calculation of radiative corrections (internal photon loops) in the corresponding non-radiative process.
- For $y = 1 + r$, where the muon reaches it maximal energy (it is also the energy of the muon in the non-radiative decay), one has:

  $$IB = IB\text{-}SD^+ = IB\text{-}SD^- = SD^- = 0$$

  Only the SD^+ term contributes to the decay rate.

6 The polarization vector of the muon

The longitudinal polarization has been calculated by Bardin and Bilen'kii, and Chen et al., but not by Bijnens et al. Chen et al. have calculated all three components of the polarization vector.

6.1 The longitudinal polarization

We will use the formulae of Bardin and Bilen'kii. The formula for the longitudinal polarization contains the same ingredients as the one for the differential rate:

$$\vec{P} = P\left(\frac{\vec{v}}{v}\right)$$

where $\vec{v}$ is the velocity of the muon and P the degree of longitudinal polarization.

$$P = -\left(\frac{\alpha}{2\;\pi}\right)\left(\frac{1}{8\;\pi}\;G^2\;m_\pi\;m_\mu^2\;f_\pi^2\right)\left(\frac{d^2W}{dxdy}\right)^{-1}\frac{\sqrt{y^2-4r}}{y}\Bigg\{PIB(x,y)$$
$$+\left(\frac{m_\pi}{f_\pi}\right)\;(m_\pi\;a)\left[\left(1+\frac{b}{a}\right)PIB\text{-}SD^+(x,y)-\left(1-\frac{b}{a}\right)PIB\text{-}SD^-(x,y)\right]$$
$$+\frac{1}{4}\left(\frac{m_\pi}{m_\mu}\right)^2\left(\frac{m_\pi}{f_\pi}\right)^2(m_\pi\;a)^2\left[\left(1+\frac{b}{a}\right)^2PSD^+(x,y)+\left(1-\frac{b}{a}\right)^2PSD^-(x,y)\right]\Bigg\}$$

where $m_\pi\;a$ is dimensionless.

We have the following expressions for the various kinematical functions:

$$PIB(x,y) = \frac{1-y+r}{x^2(x+y-1-r)} \left[x^2 + 2(1+r-x) - \frac{2xyr}{x+y-1-r}\right.$$

$$\left. +2r\, \frac{xy-2(x+y-1-r)}{y^2-4r} \left(2 - x - \frac{(1+r)x}{x+y-1-r}\right)\right]$$

$$PIB\text{-}SD^{+}(x,y) = \frac{1-y+r}{x(x+y-1-r)} \left[(1-x)(x+y-1) - r\right] \frac{y(2-y-2x)}{y^2-4r}$$

$$PIB\text{-}SD^{-}(x,y) = \frac{1-y+r}{x(x+y-1-r)} \left[-(1-x)(1-y) - r + \frac{xy-2(x+y-1-r)}{y+2r}\right]$$

$$PSD^{+}(x,y) = \left[(x+y-1-r)(x-1) + rx\right] \left[x+y-1-r-2r\, \frac{xy-2(x+y-1-r)}{y^2-4r}\right]$$

$$PSD^{-}(x,y) = (1-y+r)\left[-(1-x)(1-y) - r - 2r\, \frac{xy-2(x+y-1-r)}{y^2-4r}\right]$$

For $y = 1+r$ one has $IB = IB\text{-}SD^{+} = IB\text{-}SD^{-} = SD^{-} = 0$. The only contribution comes from the SD^{+} term, which is weighted by the square of the form factor $(m_\pi\ a) = F_V = 0.017 \pm 0.008$.

Table 1 shows the longitudinal polarization of the muon as a function of the photon energy and the muon total energy.

Table 2 shows the differential decay rate as a function of the photon energy and the muon total energy.

References

[1] *"Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay"*, TRIUMF-TWIST Technical Note No.82, P. Depommier, June 25, 2003.

[2] *"Muon polarization in the $\pi^+ \to \mu^+\nu\gamma$ decay"*, TRIUMF-TWIST Technical Note No.89, P. Depommier, January 10, 2004.

[3] *"New measurement of the $\pi \to \mu\nu\gamma$ decay"*, G. Bressi et al., Nucl. Phys. B513, 555 (1998).

[4] *"Review of Particle Physics"*, Particle Data Group, Phys. Rev. D66, 010001 (2002), see page 010001-441.

[5] *"Experimental Results on the Radiative $\pi - \mu$ Decay"*, C. Castagnoli and M. Muchnik, Phys. Rev. 112, 1779 (1958).

[6] *"The Radiative Decays $\pi(K) \to l\nu_l\gamma$"*, D.Yu. Bardin and S.M. Bilen'kii, Sov. J. Nucl. Phys. 16, 311 (1973).

[7] *"Semileptonic Kaon Decays in Chiral Perturbation Theory"*, J. Bijnens et al., LNF-92/047 (P), 5 Giugno 1992 (Laboratori Nazionali di Frascati); CERN-TH.6504/92 (CERN); BUTP-92/20 (University of Bern).

[8] *"Radiative semileptonic kaon decays"*, J. Bijnens et al., Nucl. Phys. B396, 118 (1993).

[9] *"Semileptonic kaon decays"*, J. Bijnens et al., 2nd DAPHNE Physics Handbook, 315-389.

[10] *"T-violating muon polarization in $K^+ \to \mu^+\nu\gamma$"*, C.H. Chen et al., Phys. Rev. D56, 6856 (1997).

[11] *"Particle properties"*, S. Eidelmann et al. (Particle Data Group), Phys. Lett. B592, 1 (2004).

[12] *"Review of Particle Physics"*, The American Physical Society, Phys. Rev D., 66, Part 1, 2002.

x =	0.0020	0.0445	0.0870	0.1295	0.1720	0.2145	0.2569	0.2994	0.3419	0.3844	0.4269
1.5731	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
1.5701		-0.9991	-0.9981	-0.9971	-0.9960	-0.9949	-0.9940	-0.9936	-0.9942	-0.9962	
1.5672		-0.9983	-0.9964	-0.9945	-0.9927	-0.9912	-0.9903	-0.9904	-0.9919	-0.9954	
1.5642		-0.9968	-0.9947	-0.9919	-0.9894	-0.9874	-0.9865	-0.9869	-0.9895	-0.9946	
1.5613		-0.9929	-0.9926	-0.9892	-0.9860	-0.9836	-0.9825	-0.9835	-0.9872	-0.9941	
1.5583		-0.9768	-0.9898	-0.9862	-0.9823	-0.9795	-0.9785	-0.9801	-0.9851	-0.9940	
1.5554		-0.2087	-0.9852	-0.9826	-0.9783	-0.9752	-0.9743	-0.9766	-0.9831	-0.9944	
1.5524			-0.9768	-0.9778	-0.9738	-0.9705	-0.9698	-0.9731	-0.9814	-0.9955	
1.5495			-0.9595	-0.9711	-0.9682	-0.9652	-0.9651	-0.9696	-0.9801	-0.9973	
1.5465			-0.9181	-0.9609	-0.9612	-0.9591	-0.9599	-0.9661	-0.9791		
1.5436			-0.7829	-0.9443	-0.9518	-0.9518	-0.9542	-0.9625	-0.9788		
1.5406				-0.9155	-0.9385	-0.9426	-0.9476	-0.9588	-0.9792		
1.5377				-0.8610	-0.9187	-0.9307	-0.9398	-0.9552	-0.9808		
1.5347				-0.7452	-0.8876	-0.9143	-0.9302	-0.9516	-0.9841		
1.5318				-0.4381	-0.8360	-0.8908	-0.9182	-0.9483	-0.9901		
1.5288					-0.7437	-0.8554	-0.9024	-0.9457			
1.5259					-0.5615	-0.7987	-0.8808	-0.9451			
1.5229					-0.1423	-0.7005	-0.8502	-0.9502			
1.5200						-0.5087	-0.8051	-0.9729			
1.5170						-0.0345	-0.7411				

Table 1: The longitudinal polarization of the muon as a function of photon and muon energies. Horizontally: x, photon energy; vertically: y, muon total energy; both in dimensionless units (see text). For the maximal muon energy ($y = 1.5731$) the longitudinal polarization of the muon is equal to +1 (in contrast to the longitudinal polarization in normal muon decay). This value is imposed by the Structure-Dependent SD^+ term (because all other terms do not contribute at that energy). This contribution of the SD^+ term is negligible (see next table) since the muon radiative decay is dominated by the Internal Bremsstrahlung IB term, even if this IB contribution vanishes at the maximal muon energy. In order to compare to the range of muon energies which are used in TWIST: the value $y = 1.5701$ corresponds to 95% of the maximum energy of the muon. The region $1.5701 \leq y \leq 1.5731$ contains about 5% of the total probability of the radiative decay, which is only 2.0×10^{-4} of the normal muon decay.

x =	0.0020	0.0445	0.0870	0.1295	0.1720	0.2145	0.2569	0.2994	0.3419	0.3844	0.4269
1.5731	0.53e-8	0.23e-5	0.80e-5	0.15e-4	0.23e-4	0.30e-4	0.35e-4	0.35e-4	0.31e-4	0.19e-4	0.00000
1.5701		0.80e+2	0.98e+1	0.27e+1	0.11e+1	0.51e+0	0.27e+0	0.16e+0	0.10e+0	0.71e-1	
1.5672		0.15e+3	0.19e+2	0.53e+1	0.21e+1	0.99e+0	0.54e+0	0.32e+0	0.21e+0	0.14e+0	
1.5642		0.20e+3	0.27e+2	0.78e+1	0.31e+1	0.15e+1	0.79e+0	0.47e+0	0.30e+0	0.21e+0	
1.5613		0.21e+3	0.35e+2	0.10e+2	0.40e+1	0.19e+1	0.10e+1	0.62e+0	0.40e+0	0.27e+0	
1.5583		0.16e+3	0.40e+2	0.12e+2	0.48e+1	0.23e+1	0.13e+1	0.76e+0	0.49e+0	0.33e+0	
1.5554		0.12e+2	0.44e+2	0.14e+2	0.56e+1	0.27e+1	0.15e+1	0.89e+0	0.57e+0	0.39e+0	
1.5524			0.45e+2	0.15e+2	0.63e+1	0.31e+1	0.17e+1	0.10e+1	0.66e+0	0.45e+0	
1.5495			0.43e+2	0.16e+2	0.69e+1	0.34e+1	0.19e+1	0.11e+1	0.73e+0	0.51e+0	
1.5465			0.36e+2	0.17e+2	0.74e+1	0.37e+1	0.20e+1	0.12e+1	0.81e+0		
1.5436			0.24e+2	0.17e+2	0.77e+1	0.39e+1	0.22e+1	0.13e+1	0.87e+0		
1.5406				0.16e+2	0.79e+1	0.41e+1	0.23e+1	0.14e+1	0.93e+0		
1.5377				0.15e+2	0.80e+1	0.42e+1	0.24e+1	0.15e+1	0.99e+0		
1.5347				0.12e+2	0.78e+1	0.43e+1	0.25e+1	0.16e+1	0.10e+1		
1.5318				0.87e+1	0.75e+1	0.43e+1	0.26e+1	0.16e+1	0.11e+1		
1.5288					0.69e+1	0.43e+1	0.26e+1	0.17e+1			
1.5259					0.60e+1	0.41e+1	0.26e+1	0.17e+1			
1.5229					0.48e+1	0.39e+1	0.26e+1	0.17e+1			
1.5200						0.36e+1	0.25e+1	0.17e+1			
1.5170						0.31e+1	0.24e+1				

Table 2: The differential decay rate (in arbitrary units) for the radiative muon decay as a function of photon and muon energies. Horizontally: x, photon energy; vertically: y, muon total energy; both in dimensionless units (see text). For the maximal muon energy (1.5731) the contribution of the SD^+ term is the only one which does not vanish. But it is clearly negligible. The sharp rise of the differential decay rate between $x = 1.5731$ and $x = 1.5701$ is due to the appearance of the dominant IB contribution (it vanishes for $x = 1.5731$).