TRIUMF Experiment E614
Technical Note No.100

Muon depolarization in multiple scattering

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1 Preamble

This technote was originally written in the context of other experiments (Searches for a transverse polarization of the muon in the $K^+ \to \pi^0 \mu^+ \nu$ and $K^+ \to \gamma \mu^+ \nu$ decays at KEK). It may also be useful for TWIST.

The matter was also presented at the 2001 TRIUMF Summer Institute. The essential point is the rotation of the polarization vector in an electric field.

It can be shown that, for low-energy muons, depolarization due to multiple scattering is small, but is it absolutely negligible in precision experiments like TWIST?

2 Introduction

In the E614 experiment, positive muons are brought to rest by losing energy in various materials. It is important to evaluate the amount of depolarization occuring in the slowing-down process. There are several sources of depolarization. This technote deals with depolarization due to multiple scattering. Of course this effect has to be combined with other effects, which makes the situation somewhat complicated.

Depolarization of spin 1/2 particles by electromagnetic scatterings has been studied as early as 1957 by Ford and Mullin[1] and in 1964 by Iddings et al.[2].

The present note makes use of the paper on depolarization by multiple scattering of Lyuboshits[3]. It is shown that the rotation angle θ of the polarization vector $\vec{\xi}$ of a muon moving in an electric field is proportional to the rotation angle θ_0 of the momentum of that particle. The ratio θ/θ_0 is energy-dependent (a function of $\gamma = E/m$). It has two contributions: one is related to the magnetic moment anomaly and the other is related to the Thomas precession[4] (a relativistic effect).

The polarization vector rotates in an electric field. Is that surprising? In the rest frame of the muon there is a magnetic field which is induced by the Lorentz transformation. The muon feels a magnetic field.

3 Depolarization in flight. Theory

Muon depolarization in flight has been discussed in a paper by Lyuboshits[3], for which he used the results of another paper[5].

The study is limited to the depolarization in flight. It does not consider depolarization of the positive muons following muonium formation, or depolarization of the negative muons in the cascade process after muonic atom formation. Also, one neglects more complicated processes like magnetic spin-spin interactions.

Depolarization of an ensemble of particles occurs as the result of multiple scattering of the polarized particles in the atomic field (created by the nucleus and the electrons). The reason is that the various particles of the ensemble behave differently.

The formalism used for the description of polarization has been presented in a previous TWIST technical note[6]. The polarization vector is a three-vector describing the polarization properties of an ensemble of particles. Its direction defines the state of polarization and its modulus the degree of polarization. The theory of electron and photon polarization has been presented a long time ago by Tolhoek[7] and by McMaster[8].

In order to facilitate the reading, detailed calculations are presented in appendices.

The motion of the polarization vector $\vec{\xi}$ is given by:

$$\frac{d\vec{\xi}}{dt} = \vec{\Omega} \times \vec{\xi} \tag{1}$$

The motion of the particle momentum $\vec{p}$ is given by:

$$\frac{d\vec{p}}{dt} = \vec{\Omega}_0 \times \vec{p} \tag{2}$$

In the absence of a magnetic field the two vectors $\vec{\Omega}$ and $\vec{\Omega}_0$ are colinear:

$$\vec{\Omega} = \frac{\gamma - 1}{\gamma}\left[\left(\frac{g}{2} - 1\right)(\gamma + 1) + 1\right]\vec{\Omega}_0 \tag{3}$$

The proof is given in Appendix 1.

The momentum $\vec{p}$ and the polarization vector $\vec{\xi}$ are rotated around the unit vector:

$$\vec{n} = \frac{\vec{\Omega}}{|\vec{\Omega}|} = \frac{\vec{\Omega}_0}{|\vec{\Omega}_0|} \tag{4}$$

by angles θ_0 and θ, respectively. And:

$$\theta = \frac{\gamma - 1}{\gamma}\left[\left(\frac{g}{2} - 1\right)(\gamma + 1) + 1\right]\theta_0 \tag{5}$$

There are two contributions: the first term is related to the magnetic moment anomaly, the second one to the Thomas precession[4].

Figures 1 and 2 show the variation of the ratio θ/θ_0 vs. γ. For $\gamma \sim 1$ (small kinetic energy) θ is close to zero. For $\gamma \to \infty$ the ratio θ/θ_0 increases linearly. The second graph corresponds to low-energy muons. The cross corresponds to TWIST muons.

It can be shown that if θ is small the polarization vector is rotated by the angle $\overline{\theta}$:

$$\overline{\theta} = \theta \sin\psi \tag{6}$$

where ψ is the angle between the initial polarization vector and the vector $\vec{n}$. The proof is given in Appendix 2.

We now proceed to the calculation of the depolarization factor and consider first the case of longitudinal polarization:

$$\vec{\xi}_0 = \frac{\vec{p}_0}{|\vec{p}_0|} \tag{7}$$

This corresponds to:

$$\psi = \frac{\pi}{2} \qquad \sin\psi = 1 \qquad \overline{\theta} = \theta \tag{8}$$

Multiple scattering is rotational-invariant around the direction of $\vec{p}_0$ (or $\vec{\xi}_0$), therefore the mean values $< \vec{p} >$ and $< \vec{\xi} >$ are colinear with $\vec{p}_0$ (or $\vec{\xi}_0$). For individual particles the longitudinal and transverse components of the polarization vector are given by:

$$\xi_L = \cos\theta = \cos\overline{\theta} \qquad \xi_T = \sin\theta = \sin\overline{\theta} \tag{9}$$

Averaging over the ensemble:

$$< \xi_L > = < \cos\overline{\theta} > \qquad < \xi_T > = < \sin\overline{\theta} > = 0 \tag{10}$$

For small angles:

$$< \xi_L > = 1 - \frac{1}{2} < (\overline{\theta})^2 > \tag{11}$$

The modulus of the polarization vector (degree of polarization) is:

$$< \xi_L > = < \cos\overline{\theta} > \tag{12}$$

And the variation of the degree of polarization (depolarization factor):

$$\eta_L = 1 - < \xi_L > = 1 - < \cos\overline{\theta} > \tag{13}$$

For small angles:

$$\eta_L = 1 - < \xi_L > = \frac{1}{2} < (\overline{\theta})^2 > \tag{14}$$

For a muon:

$$\frac{g}{2} - 1 = 1.16 \times 10^{-3} \tag{15}$$

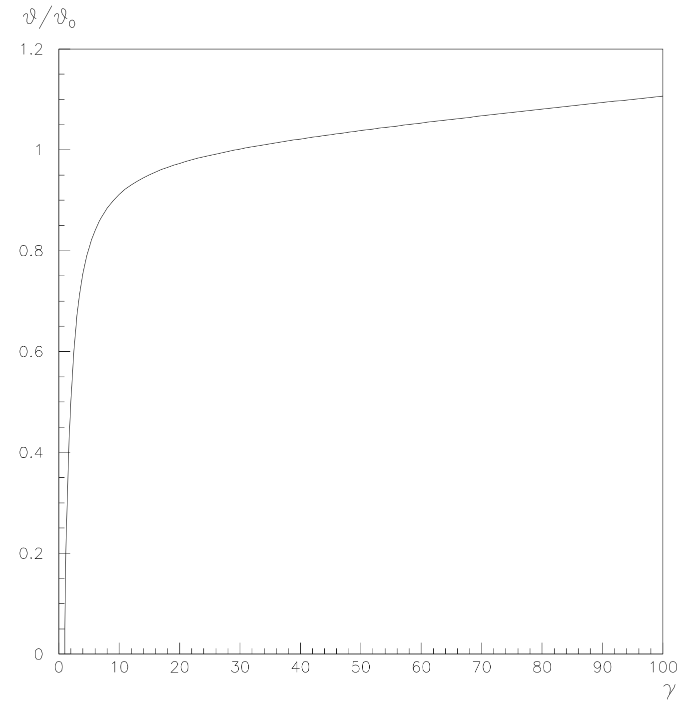

Figure 1: The ratio θ/θ_0 vs. γ.

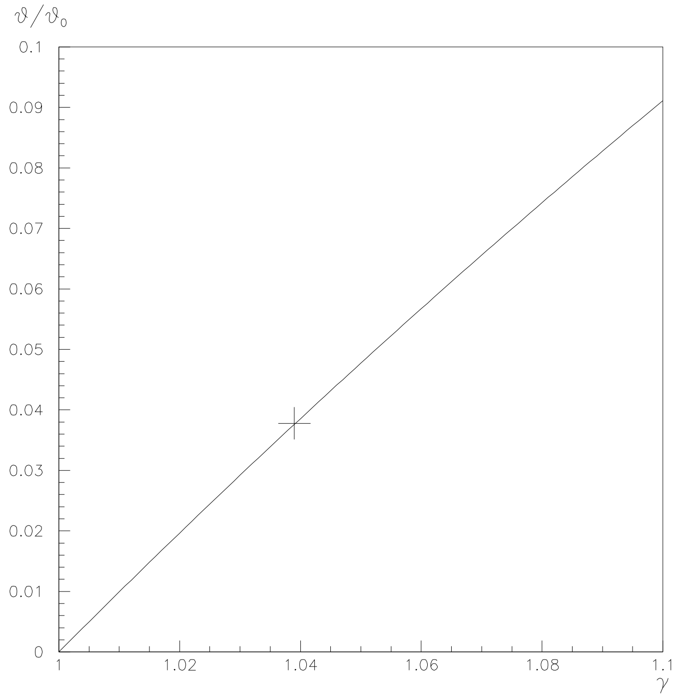

Figure 2: The ratio θ/θ_0 vs. γ for small values of γ.

For a muon of kinetic energy < 106 MeV, $\gamma < 2$. The first term in the square brackets of Equations 3 and 5 is $< 3.5 \times 10^{-3}$, and can be neglected:

$$\theta = \frac{\gamma - 1}{\gamma}\, \theta_0 \tag{16}$$

The root mean square angle of deflection of the polarization vector is related to the root mean square angle of multiple Coulomb scattering by the relation:

$$< \theta^2 > = \left(\frac{\gamma - 1}{\gamma}\right)^2 < \theta_0^2 > \tag{17}$$

The multiple scattering angle is given by Molière's theory:

$$< \theta_0^2 > = z^2 \left(\frac{M_s}{m}\right)^2 \left(\frac{\gamma}{\gamma^2 - 1}\right)^2 \frac{\Delta l}{L_{rad}} \tag{18}$$

Therefore:

$$< \theta^2 > = z^2 \left(\frac{M_s}{m}\right)^2 \left(\frac{1}{\gamma + 1}\right)^2 \frac{\Delta l}{L_{rad}} = z^2 \left(\frac{M_s}{m}\right)^2 \left(\frac{m}{T + 2m}\right)^2 \frac{\Delta l}{L_{rad}} \tag{19}$$

where m is the mass of the particle, T its kinetic energy, Δl the thickness of the material and L_{rad} its radiation length. M_s is a constant[3].

Finally, the depolarization factor for the longitudinal polarization is given by:

$$\eta_L = 1 - \cos \overline{\theta} = \frac{1}{2} < \theta^2 > \tag{20}$$

$$\eta_L = \frac{1}{2} z^2 \left(\frac{M_s}{m}\right)^2 \left(\frac{m}{T + 2m}\right)^2 \frac{\Delta l}{L_{rad}} \tag{21}$$

For the transverse polarization there is an additional factor of one-half[3].

4 Numerical calculation

Let us consider the longitudinal polarization.

We have $z = 1$, $m = 106$ MeV, $M_s = 21$ MeV[3]. Therefore:

$$\eta_L = 5.0 \times 10^{-3} \left(\frac{1}{1 + 0.0047T}\right)^2 \frac{\Delta l}{L_{rad}} \tag{22}$$

where T is in MeV.

As an example, Let us consider a muon of 106 MeV initial kinetic energy which is brought to rest. To get the total depolarization the above expression has to be integrated over the range of the particle.

$$\eta_L = 5.0 \times 10^{-3} \frac{R}{L_{rad}} \int_0^1 \left(\frac{1}{1 + 0.0047T(x)}\right)^2 dx \tag{23}$$

The energy dependent term varies from 0.44 to 1.0. A rough estimate is obtained by taking the average, 0.72.

$$\eta_L = 3.6 \times 10^{-3} \frac{R}{L_{rad}} \tag{24}$$

An alternative calculation is:

$$\eta_L = 5.0 \times 10^{-3} \frac{1}{L_{rad}} \int_0^{T_i} \left(\frac{1}{1 + 0.0047T} \right)^2 \left| \frac{dT}{dx} \right|^{-1} dT \tag{25}$$

where T_i is the initial kinetic energy (106 MeV).

Assuming a constant value for dT/dx:

$$\frac{dT}{dx} = \frac{T_i}{R} \tag{26}$$

one gets:

$$\eta_L = 5.0 \times 10^{-3} \frac{1}{L_{rad}} \left| \frac{dT}{dx} \right|^{-1} \int_0^{T_i} \left(\frac{1}{1 + 0.0047T} \right)^2 dT$$

$$= 5.0 \times 10^{-3} \frac{R}{L_{rad}} \frac{1}{1 + 0.0047T_i} = 3.3 \times 10^{-3} \frac{R}{L_{rad}} \tag{27}$$

which is almost the same result as before.

Using the tables of the Particle Data Group[9] to find the range, we get the following results (Table 1), using an average of the two methods. It should be noted that the determination of the range was done by looking at the graphs given in reference[9], therefore these results may be inaccurate.

Medium	Pb	Al
Density (g-cm^{-3})	11.35	2.70
Radiation length (g-cm^{-2})	6.37	24.01
Muon range (106 MeV, 184 MeV/c) (g-cm^{-2})	70.	47.
Muon range (53 MeV, 119 MeV/c) (g-cm^{-2})	20.	12.
Muon range (10 MeV, 47 MeV/c) (g-cm^{-2})	1.5	0.95
Depolarization (106 MeV, longitudinal) (%)	3.8	0.7
Depolarization (53 MeV, longitudinal) (%)	1.3	0.2
Depolarization (10 MeV, longitudinal) (%)	0.1	0.02

Table 1: Depolarization factors (longitudinal polarization) for two materials and several initial muon kinetic energies (crude evaluation).

To obtain a better result a numerical integration has been performed, using the Bethe-Bloch formula for the rate of energy loss[9]. The accuracy of this formula is sufficient for our purpose.

$$
\begin{aligned}
-\frac{dE}{dx} &= 4\pi N_A r_e^2 m_e c^2 z^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\mathrm{Log}\left(\frac{2m_e c^2\beta^2\gamma^2 T_{max}}{I^2}\right)^{1/2} - \beta^2 - \frac{\delta}{2}\right] \\
&= 0.3071(\mathrm{MeV\ cm^2\ g^{-1}}) z^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\mathrm{Log}\left(\frac{2m_e c^2\beta^2\gamma^2 T_{max}}{I^2}\right)^{1/2} - \beta^2 - \frac{\delta}{2}\right] \qquad (28)
\end{aligned}
$$

With the approximation:

$$
T_{max} = 2m_e c^2\beta^2\gamma^2 \qquad (29)
$$

we have:

$$
-\frac{dE}{dx} = 0.3071(\mathrm{MeV\ cm^2\ g^{-1}})\ z^2 \frac{Z}{A}\frac{1}{\beta^2}\left[\mathrm{Log}\left(\frac{2m_e c^2\beta^2\gamma^2}{I}\right) - \beta^2 - \frac{\delta}{2}\right] \qquad (30)
$$

The ionisation constant I has been calculated according to the formula[9]:

$$
I = 16Z^{0.9}\ \mathrm{eV} \qquad (31)
$$

and δ has been neglected.

The integration on the energy is performed from a low-energy cut-off T_0 to T_i, the initial energy. The results are insensitive to the value of T_0 below 1 MeV. Also, the accuracy of the numerical integration has been tested. The results are shown in Table 2.

Medium	Pb	Al
Depolarization (106 MeV, longitudinal) (%)	3.10	0.54
Depolarization (53 MeV, longitudinal) (%)	1.40	0.24
Depolarization (10 MeV, longitudinal) (%)	0.11	0.018

Table 2: Depolarization factors (longitudinal polarization) for various materials and various initial muon kinetic energies (numerical integration).

For the transverse polarization there is an additional factor of 1/2[3]. The results are shown in Table 3.

Medium	Pb	Al
Depolarization (106 MeV, transverse) (%)	1.55	0.27
Depolarization (53 MeV, transverse) (%)	0.70	0.12
Depolarization (10 MeV, transverse) (%)	0.055	0.009

Table 3: Depolarization factors (transverse polarization) for various materials and various initial muon kinetic energies.

Chiba and Yang[10] have performed a simulation and get the following values for the longitudinal depolarization in lead: 1.5% and 0.1% for 53 and 10 MeV, respectively. This is in reasonable agreement with the values given above: 1.40% and 0.11%.

Finally, we present in Tables 4 and 5 the calculated ranges and depolarization factors for Pb and Al respectively, as a function of initial muon energy T_i.

T_i (MeV)	Range (g-cm^{-2})	Range (cm)	Depolarization (%)
10	1.50	0.13	0.11
20	4.87	0.43	0.34
30	9.54	0.84	0.63
40	15.1	1.34	0.96
50	21.5	1.89	1.30
60	28.4	2.50	1.64
70	35.7	3.15	1.98
80	43.4	3.82	2.31
90	51.3	4.52	2.62
100	59.4	5.24	2.93
110	67.7	5.97	3.22
120	76.2	6.71	3.50
130	84.7	7.47	3.77
140	93.4	8.23	4.02
150	102.	8.99	4.26
160	111.	9.77	4.49
170	120.	10.5	4.71
180	128.	11.3	4.92
190	137.	12.1	5.12
200	146.	12.9	5.31

Table 4: Pb - Range and depolarization factor (longitudinal polarization).

T_i (MeV)	Range (g-cm^{-2})	Range (cm)	Depolarization (%)
10	0.91	0.34	0.018
20	3.05	1.13	0.057
30	6.05	2.24	0.11
40	9.70	3.59	0.16
50	13.9	5.13	0.22
60	18.4	6.81	0.28
70	23.2	8.60	0.34
80	28.3	10.5	0.40
90	33.6	12.4	0.45
100	39.0	14.5	0.51
110	44.6	16.5	0.56
120	50.3	18.6	0.61
130	56.0	20.7	0.66
140	61.8	22.9	0.70
150	67.7	25.1	0.75
160	73.6	27.3	0.79
170	79.6	29.5	0.83
180	85.6	31.7	0.86
190	91.6	33.9	0.90
200	97.6	36.1	0.93

Table 5: Al - Range and depolarization factor (longitudinal depolarization).

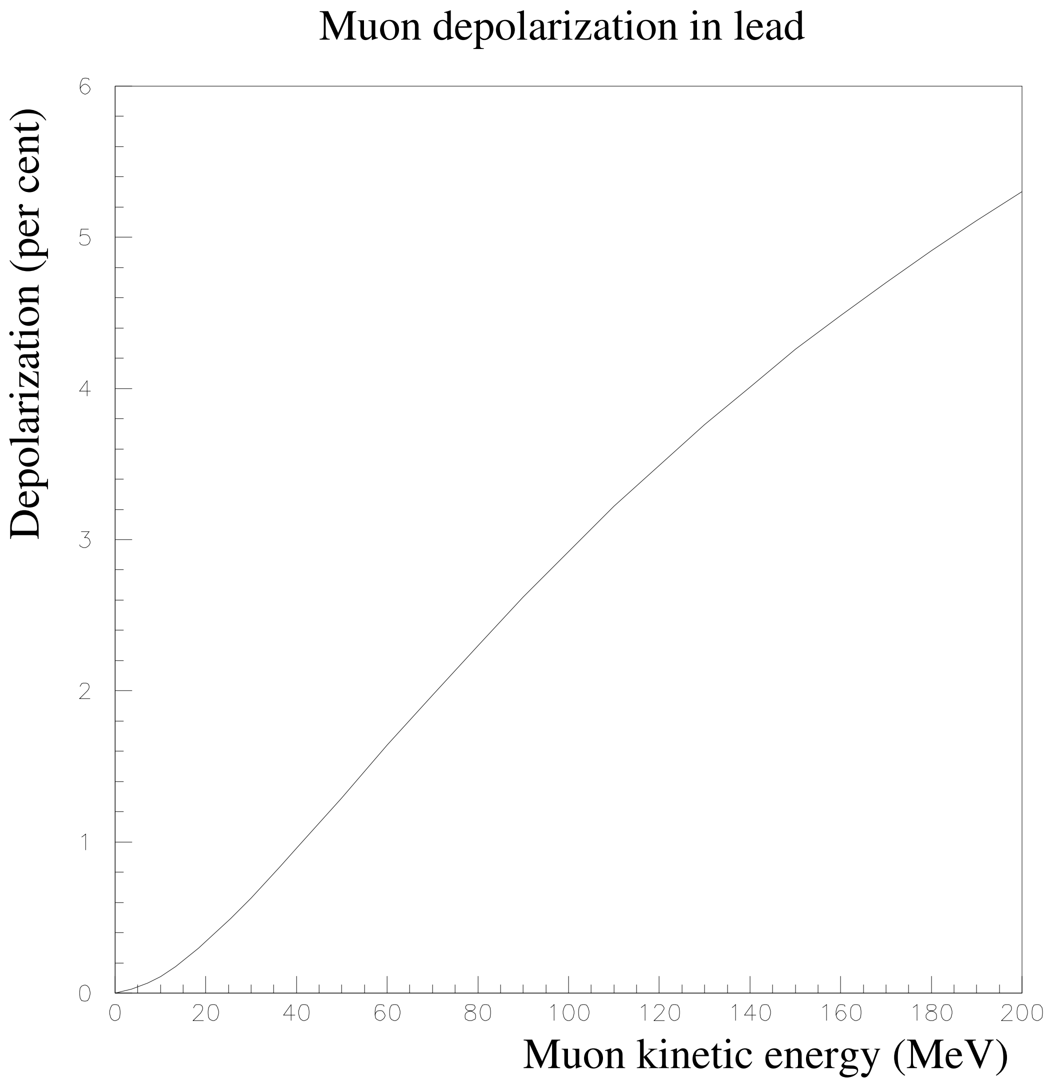

Figure 3: Depolarization factor in lead (longitudinal polarization), using Equation 25.

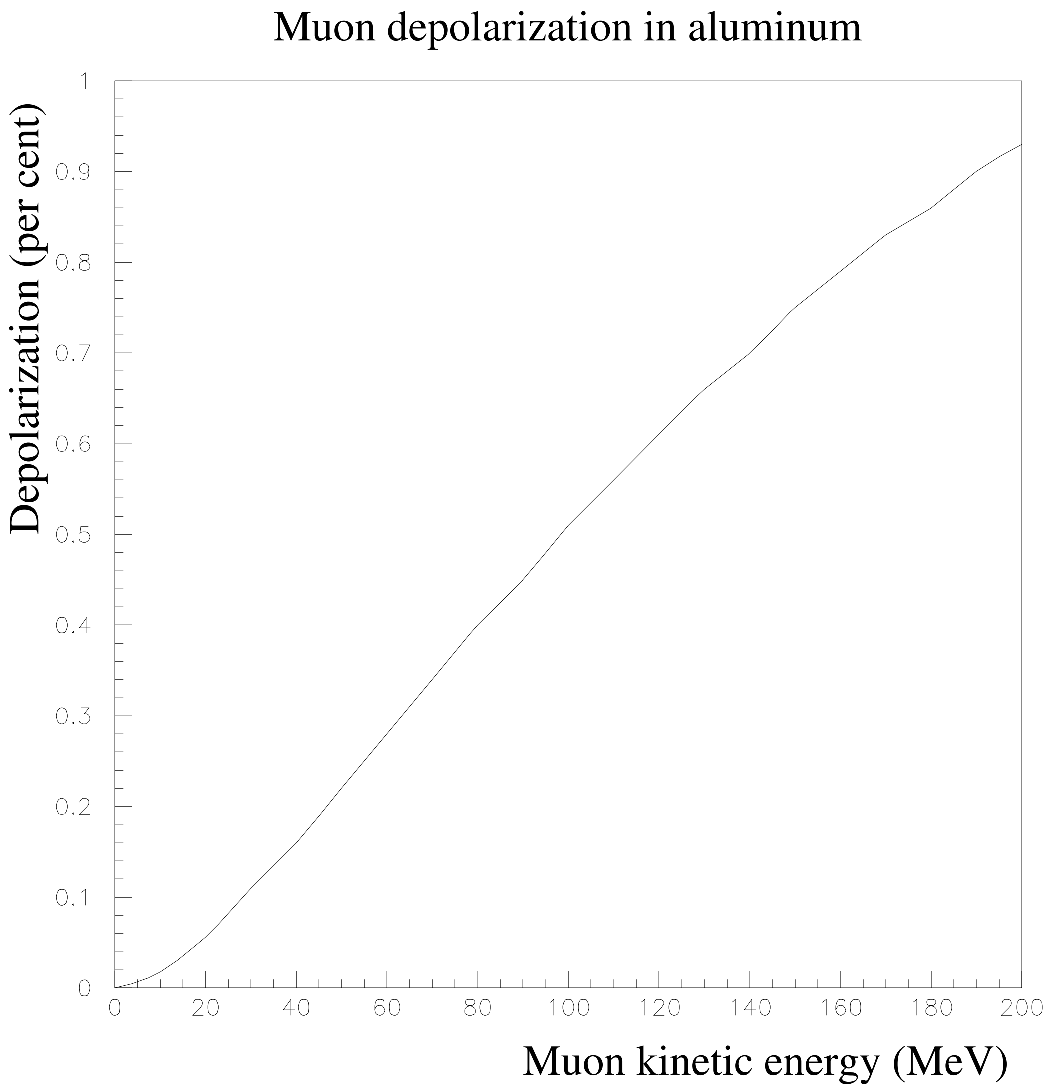

Figure 4: Depolarization factor in aluminum (longitudinal polarization), using Equation 25.

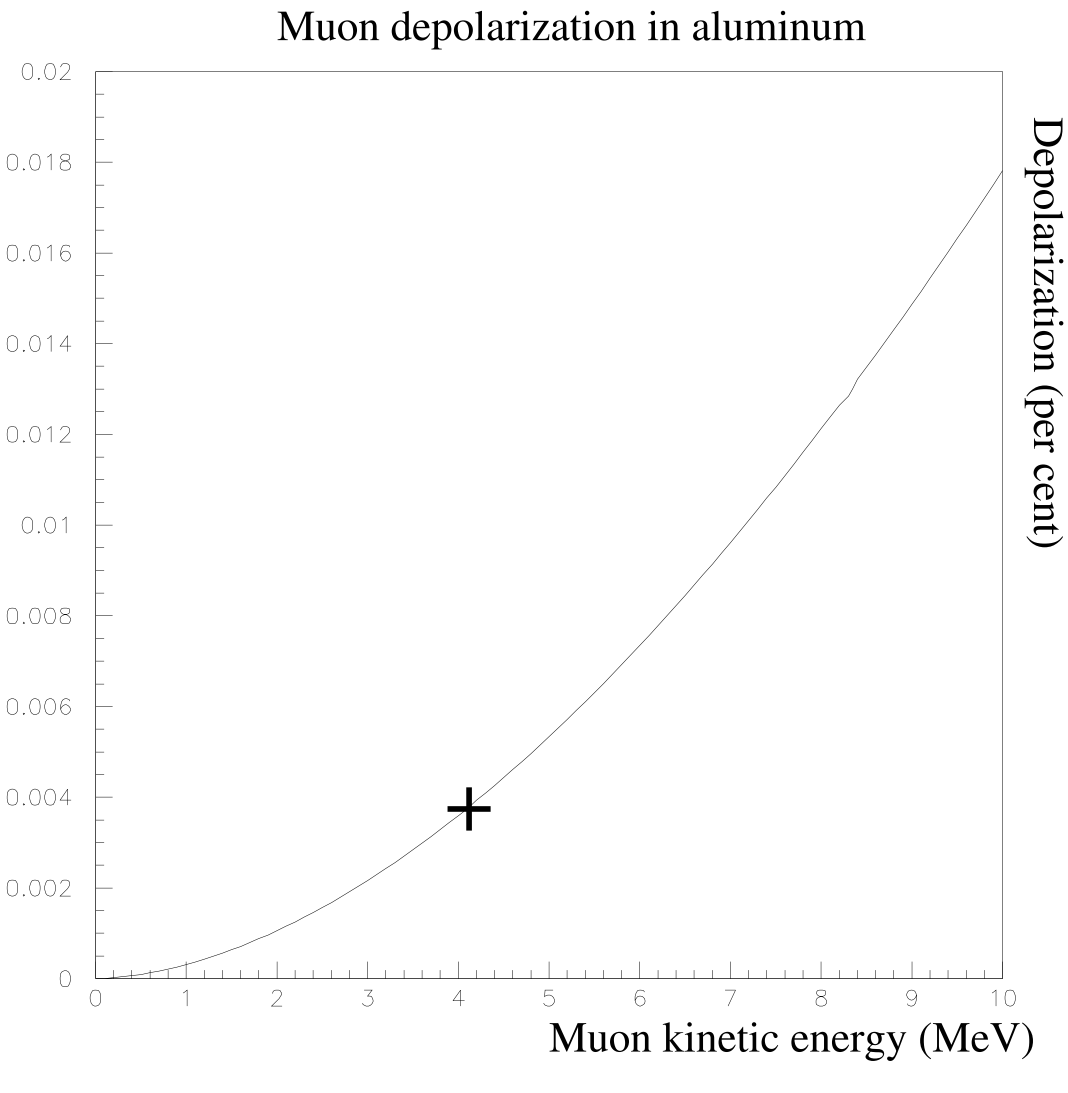

Figure 5: Depolarization factor in aluminum (longitudinal polarization), using Equation 25.

5 Low-energy muons

What about low-energy muons? Assuming that the theory is still applicable at low energies, a calculation for 4 MeV muons yields a depolarization factor $\sim 0.004\%$ (4×10^{-5}). Results are shown on Figure 5. This may not be absolutely negligible in TWIST.

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6 Appendix 1

The two basic ingredients of the calculation are:

- The Bargmann-Michel-Telegdi equation[11], which is written as[6]:

$$\frac{d\vec{\xi}}{dt} = \vec{\Omega} \times \vec{\xi} \tag{32}$$

with:

$$\vec{\Omega} = -\frac{e}{2mc} g \left[\vec{\mathcal{B}} - \frac{\gamma - 1}{\gamma}\ \vec{n}\ (\vec{\mathcal{B}} \cdot \vec{n}) + \left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c} \right) \right] - (\gamma - 1) \left(\vec{n} \times \frac{d\vec{n}}{dt} \right) \tag{33}$$

As usual, $\vec{\xi}$ is the polarization vector (a three-vector) expressed in the "instantaneous" rest frame of the particle, but all the kinematics are done in the laboratory system: t is the time in the laboratory frame, and so is the velocity ($\vec{v}$) and the electromagnetic field ($\vec{E}$, $\vec{H}$). The other definitions are:

$$\gamma = (1 - \beta^2)^{-1/2} \qquad \text{(Lorentz factor)} \tag{34}$$

$$\vec{n} = \frac{\vec{v}}{v} = \frac{\vec{p}}{p} \qquad \text{(a unit vector along the particle momentum)} \tag{35}$$

and g (gyromagnetic ratio) is defined by the relation between the magnetic moment $\vec{\mu}$ and the spin $\vec{S}$:

$$\vec{\mu} = \frac{eg}{2mc} \hbar \vec{S} \tag{36}$$

It should be noted that the magnetic field in the rest frame of the particule is given by:

$$\vec{\mathcal{B}}^* = \gamma \left[\vec{\mathcal{B}} - \frac{\gamma - 1}{\gamma}\ \vec{n}\ (\vec{\mathcal{B}} \cdot \vec{n}) + \left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c} \right) \right] = \vec{\mathcal{B}}_L + \gamma \vec{\mathcal{B}}_T + \gamma \left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c} \right) \tag{37}$$

where $\vec{\mathcal{B}}_L$ and $\vec{\mathcal{B}}_T$ are the longitudinal and transverse components of the magnetic field in the laboratory system, respectively.

Therefore the interpretation of the first term in Equation 33 is clear:

$$\vec{\Omega}_1 = -\frac{e}{2mc\gamma}\ g\ \vec{\mathcal{B}}^* \tag{38}$$

The second term is a relativistic effect known as the Thomas precession[4]:

$$\vec{\Omega}_2 = -(\gamma - 1) \left(\vec{n} \times \frac{d\vec{n}}{dt} \right) \tag{39}$$

This term is proportional to the kinetic energy of the particle and becomes important for high-energy particles.

- The equation of motion:

$$\frac{d\vec{p}}{dt} = e\vec{\mathcal{E}} + \frac{1}{c}(\vec{v} \times \vec{\mathcal{B}}) \tag{40}$$

which can be written as:

$$\frac{d\vec{p}}{dt} = \vec{\Omega}_0 \times \vec{p} \tag{41}$$

One has:

$$\vec{p} \times \frac{d\vec{p}}{dt} = \vec{p} \times \left(\vec{\Omega}_0 \times \vec{p}\right) = p^2\vec{\Omega}_0 - (\vec{p} \cdot \vec{\Omega}_0)\vec{p} \tag{42}$$

But $\vec{\Omega}_0$ can be chosen perpendicular to $\vec{p}$. Therefore:

$$\vec{\Omega}_0 = \frac{1}{p^2}\left(\vec{p} \times \frac{d\vec{p}}{dt}\right) \tag{43}$$

But:

$$\vec{n} \times \frac{d\vec{n}}{dt} = \frac{\vec{p}}{p} \times \frac{d}{dt}\left(\frac{\vec{p}}{p}\right) = \frac{\vec{p}}{p} \times \left[\frac{1}{p}\frac{d\vec{p}}{dt} - \frac{1}{p^2}\,\vec{p}\right] = \frac{1}{p^2}\left(\vec{p} \times \frac{d\vec{p}}{dt}\right) \tag{44}$$

Therefore:

$$\vec{\Omega}_0 = \vec{n} \times \frac{d\vec{n}}{dt} \tag{45}$$

Using a previous formula:

$$\vec{\Omega}_0 = \frac{1}{p^2}\left(\vec{p} \times \frac{d\vec{p}}{dt}\right) = \frac{e}{p^2}\,\vec{p} \times \left[\vec{\mathcal{E}} + \frac{1}{c}(\vec{v} \times \vec{\mathcal{B}})\right] \tag{46}$$

After some calculations:

$$\vec{\Omega}_0 = -\frac{e}{mc\gamma}\left[\vec{\mathcal{B}} - \vec{n}(\vec{\mathcal{B}} \cdot \vec{n}) + \frac{\gamma^2}{\gamma^2 - 1}\left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c}\right)\right] \tag{47}$$

In the absence of a magnetic field in the laboratory system:

$$\vec{\Omega} = -\frac{e}{2mc}g\left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c}\right) - (\gamma - 1)\left(\vec{n} \times \frac{d\vec{n}}{dt}\right) \tag{48}$$

$$\vec{\Omega}_0 = -\frac{e}{mc}\,\frac{\gamma}{\gamma^2 - 1}\left(\vec{\mathcal{E}} \times \frac{\vec{v}}{c}\right) \tag{49}$$

Therefore the two vectors $\vec{\Omega}$ and $\vec{\Omega}_0$ are colinear:

$$\vec{\Omega} = \left[\left(\frac{g}{2} - 1\right)\frac{\gamma^2 - 1}{\gamma} + \frac{\gamma - 1}{\gamma}\right]\vec{\Omega}_0 = \frac{\gamma - 1}{\gamma}\left[\left(\frac{g}{2} - 1\right)(\gamma + 1) + 1\right]\vec{\Omega}_0 \tag{50}$$

The first term is dominant for high-energy particles.

7 Appendix 2

Let us make the following choice for the vectors:

$$\vec{p}_0 \quad \begin{pmatrix} p_0 \\ 0 \\ 0 \end{pmatrix} \qquad\qquad \vec{\xi}_0 \quad \begin{pmatrix} \sin\psi\cos\phi \\ \sin\psi\sin\phi \\ \cos\psi \end{pmatrix} \tag{51}$$

And a rotation of angle θ around the z axis:

$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{52}$$

Then the momentum and the polarization vector are rotated:

$$\vec{p}_1 = R_z(\theta)\vec{p}_0 \qquad \vec{\xi}_1 = R_z(\overline{\theta})\vec{\xi}_0 \tag{53}$$

We have for these two vectors:

$$\vec{p}_1 \quad \begin{pmatrix} p_0\cos\theta \\ p_0\sin\theta \\ 0 \end{pmatrix} \qquad\qquad \vec{\xi}_1 \quad \begin{pmatrix} \cos\theta\sin\psi\cos\phi + \sin\theta\sin\psi\sin\phi \\ -\sin\theta\sin\psi\cos\phi + \cos\theta\sin\psi\sin\phi \\ \cos\psi \end{pmatrix} \tag{54}$$

With:

$$\cos\overline{\theta} = (\vec{\xi}_0 \cdot \vec{\xi}_1) \tag{55}$$

we have:

$$\begin{aligned} \cos\overline{\theta} &= \sin\psi\cos\phi(\cos\theta\sin\psi\cos\phi + \sin\theta\sin\psi\sin\phi) \\ &+ \sin\psi\sin\phi(-\sin\theta\sin\psi\cos\phi + \cos\theta\sin\psi\sin\phi) + \cos^2\psi \\ &= \cos\theta\sin^2\psi + \cos^2\psi \end{aligned} \tag{56}$$

If the angles θ and $\overline{\theta}$ are small:

$$1 - \frac{1}{2}(\overline{\theta})^2 = \left(1 - \frac{1}{2}\theta^2\right)\sin^2\psi + \cos^2\psi = 1 - \frac{1}{2}\theta^2\sin^2\psi \tag{57}$$

Hence:

$$\overline{\theta} = \theta\sin\psi \tag{58}$$

8 Appendix 3

As a check of the calculations, it is interesting to try reproducing the graph presented by Lyuboshits (page 364 of reference [3]). Here the depolarization factor (for longitudinal polarization) is plotted versus the dimensionless variable p_μ/mc. Our results are presented in Table 6 and Figure 6. The graph in Figure 7 is taken from the paper of Lyuboshits. Numerical values have been obtained from the graph, therefore they are not very accurate.

There is a slight difference ($\sim 3\%$) for η_L at the maximum value of p_μ/c, our result being slightly different smaller that of Lyuboshits. This could be easily explained by a difference in the calculation of dE/dx. Figure 8 shows the comparison.

p_μ/mc	T (MeV)	η_L
0.0	0.00	0.000
0.1	0.53	0.001
0.2	2.09	0.008
0.3	4.66	0.031
0.4	8.15	0.079
0.5	12.5	0.160
0.6	17.6	0.278
0.7	23.3	0.434
0.8	29.7	0.623
0.9	36.5	0.843
1.0	43.8	1.09
2.0	131.	3.78
3.0	229.	5.80
4.0	330.	7.12
5.0	434.	8.02
6.0	538.	8.66
7.0	642.	9.14
8.0	747.	9.50
9.0	852.	9.79
10.0	957.	10.0
11.0	1062.	10.2
12.0	1168.	10.4
13.0	1273.	10.5
14.0	1379.	10.6

Table 6: Depolarization factor in lead (longitudinal polarization), using Equation 25.

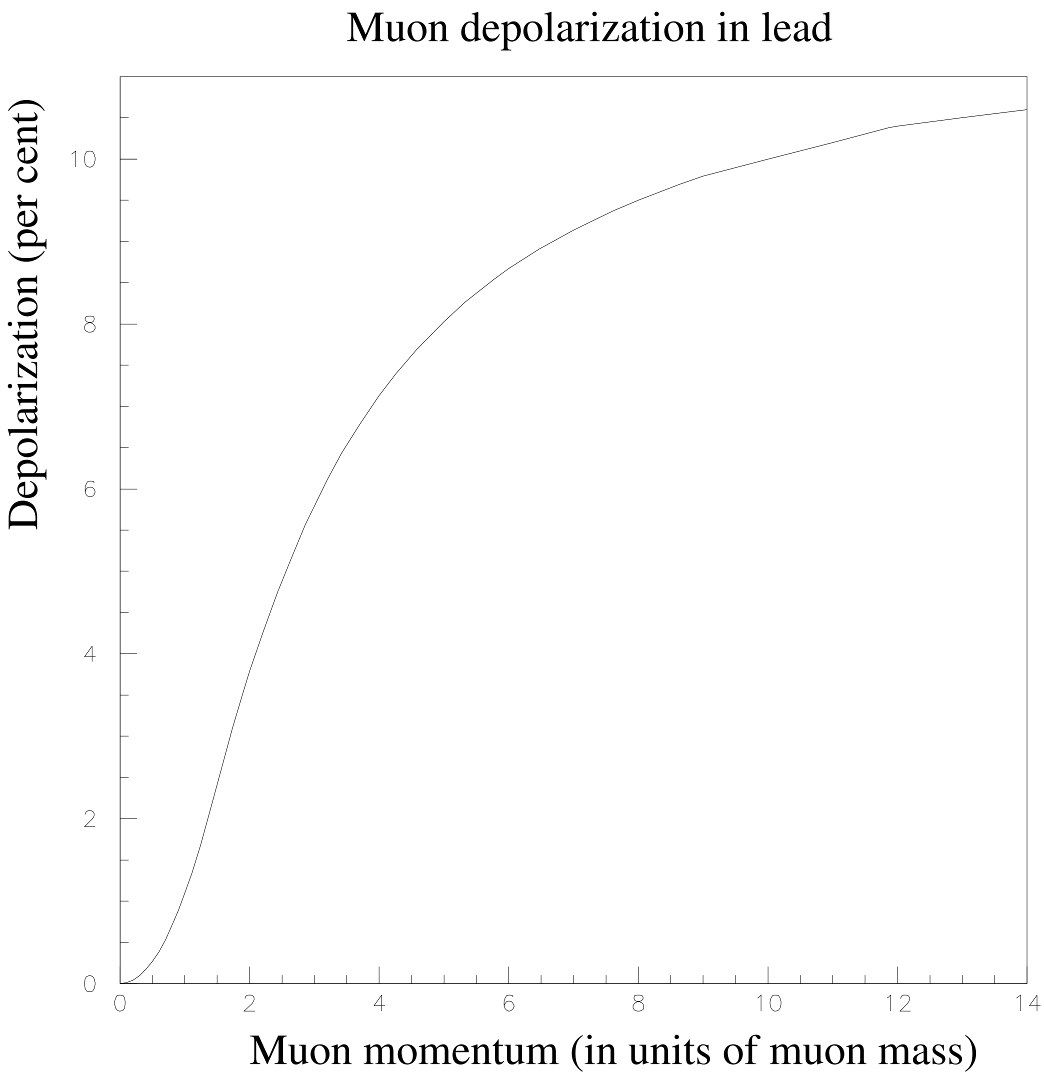

Figure 6: Depolarization factor in lead (longitudinal polarization), using Equation 25.

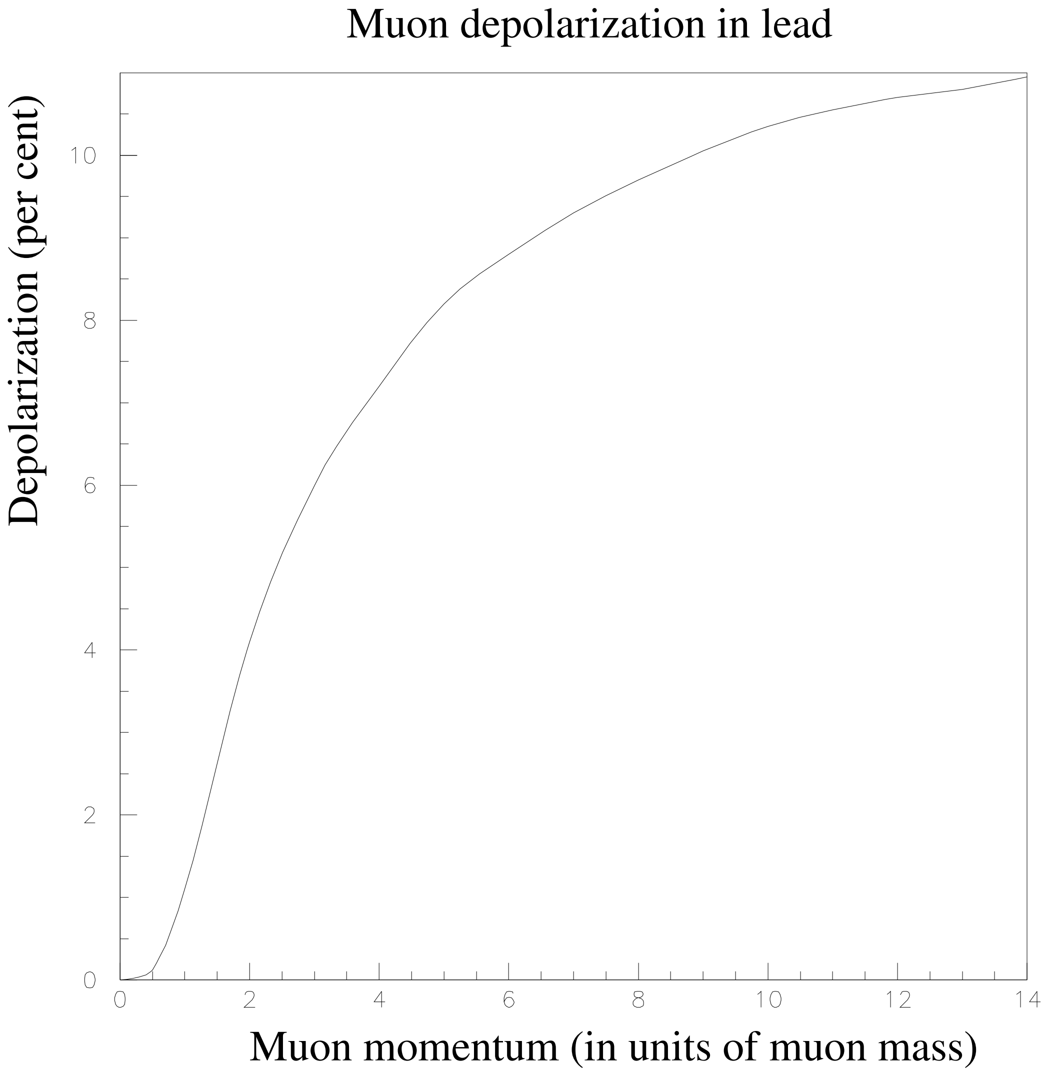

Figure 7: Depolarization factor in lead (longitudinal polarization), using the graph given by Lyuboshits.

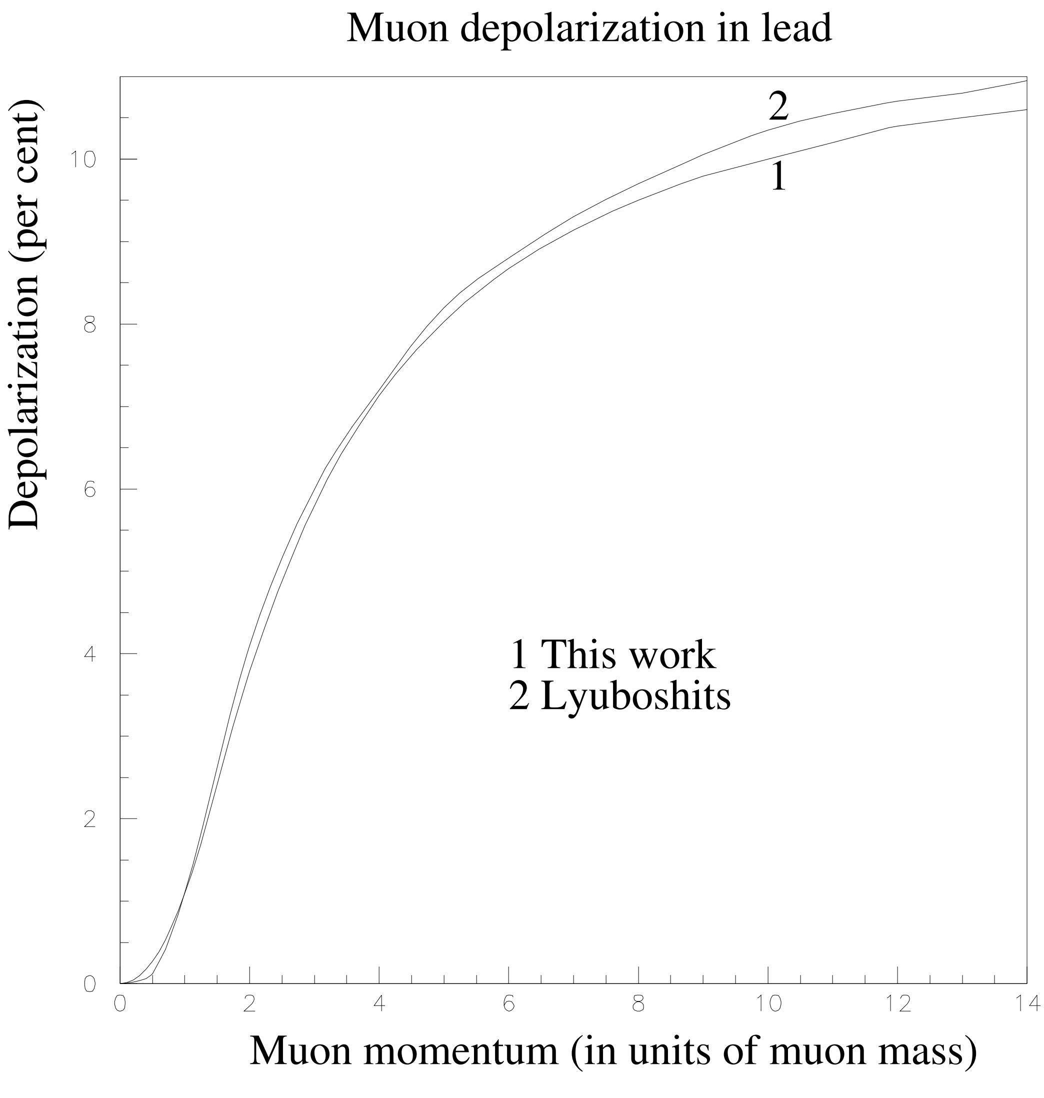

Figure 8: Depolarization factor in lead (longitudinal polarization). A comparison.